MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXXIV. SUPPLEMENTARY NUMBER. No. 9

The Radiative Equilibrium of a Rotating System of Gaseous Masses.

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1. *Introduction and Summary.*—A rotating system of gaseous masses is said to be in mechanical equilibrium if the force of gravity, resulting from the force of attraction and the centrifugal force, is exactly balanced by the force of the total pressure, composed of the gas pressure and the radiation pressure. A system of gaseous bodies of given masses, mean densities, and angular velocity may be in mechanical equilibrium in an infinite number of different ways. Every such equilibrium is characterised by a functional relation between density and gravity-potential or, if one prefer, of a functional connection,

$$(1) \quad . \quad . \quad . \quad . \quad . \quad p = f(T)$$

between the gas pressure p, and the temperature T. But the function f may be arbitrary.

Through Schwarzschild's* introduction of the idea of radiative equilibrium an important step towards the solution of the problem was taken.

Schwarzschild† derived the differential equation for radiative intensity in a gaseous mass. Schwarzschild's equation fully defines the intensity of radiation, and thence also the absorption, for a given density, temperature, and absorption-coefficient of the gas.

Eddington‡ went further and derived the differential equation for radiative equilibrium. For the difference between emission and absorption, Eddington obtained an approximate expression, not applicable very near the boundary surface of the gaseous mass, taking into account density,

* Göttingen, *Nachr. v. d. K. Ges. d. Wiss.*, 1906, p. 41; *Sitz-berichte d. K. preuss. Akad. d. Wiss.*, 28, 1914, p. 1183.

† *Loc. cit.*

‡ *M.N.*, 77, 20 (1916); *Zeitschr. f. Physik*, 7, 351 (1921).

temperature, and absorption-coefficient and their derivatives. In the state of radiative equilibrium this difference or defect is compensated by the liberation of new energy. But the liberation of energy as well as the absorption-coefficient depends physically on the nature (composition) and condition or state (density and temperature) of the gas. In radiative equilibrium, therefore, density and temperature must satisfy a partial differential equation including the production of energy, and the coefficient of absorption, as functions of the nature and condition of the gas. But the former function is at present completely unknown. For the latter, different theories give different expressions. It may therefore be said that Eddington introduced a new differential equation, but also one or two unknown functions, in the problem.

In Eddington's theory for the radiative equilibrium of stars without rotation, it is assumed that the production of energy and the absorption-coefficient are constant within the star. On the basis of this hypothesis Eddington arrives at the adiabatic law

$$(2) \quad . \quad . \quad . \quad . \quad . \quad p = A \,.\, T^4,$$

where A is constant within the star. Eddington shows that even a rather large change in the production of energy within the star has only little influence on the state of equilibrium.

A. Kohlschütter,* on the other hand, assumes that the production of energy is proportional to the temperature. In his theory the functional relation (1) is more complicated.

Eddington's theory has been generalised by Milne † to rotating stars. Milne starts, like Eddington, with the assumption that the production of energy is constant within the star. This assumption, which in Eddington's theory led to the very simple adiabatic law (2), in Milne's theory no longer has such simple consequences.

In the following theory for the radiative equilibrium of a rotating system of gaseous masses, it is assumed as a *fundamental hypothesis that the nature of the gas is constant over every level surface,* but may change from one level surface to another. It is proved that the state of the gas is then constant along a level surface. Then also the production of energy, the coefficient of absorption, and the mean molecular weight are constant on a level surface. It may be remarked that our fundamental hypothesis attributes to the gaseous mass a property which will continue during its development when once existing.

A consequence of our general fundamental hypothesis is that the production of energy $4\pi\epsilon$ per second and gram within the gaseous mass follows the simple law

$$(3) \quad . \quad . \quad . \quad . \quad 4\pi\epsilon = B\left(1 - \frac{\omega^2}{2\pi G\rho}\right),$$

where ω is the angular velocity, ρ the density, and G the constant of attraction, while B is constant within the gaseous mass. If in radiative

* *Publ. d. Astrophys. Obs. zu Potsdam,* Bd. **25**, No. 78 (1922).
† *M.N.*, **83**, 118 (1923).

equilibrium the nature of the gas is constant on a level surface, then the gaseous mass must be so arranged that its production of energy follows the law (3).

When $\omega = 0$, ϵ is constant, as in Eddington's theory. On the other hand, ϵ cannot be constant within a rotating gaseous mass in radiative equilibrium unless the density is constant also. The proof of this statement depends only on the postulate that the nature of the gas is constant on a level surface.

As a consequence of the law (3) regulating the production of energy, a general expression for the total radiation of every partial body of the rotating system is deduced.

It is also shown that the net flux of energy per second and cm.^{2} through a level surface anywhere within the partial body is proportional to the gravity. Where the boundary surface of the body has an edge or a point through which the gas can escape, the intensity of radiation is zero. This is perhaps the explanation of the dark equatorial belt in certain lenticular nebulæ.

According to modern theories of emission, the coefficient of absorption follows at least approximately the law * $k = \mathrm{K}p\mathrm{T}^{-4}$, where K is a constant. We accept a more general law

$$(4) \quad . \quad . \quad . \quad . \quad . \quad k = k(p\mathrm{T}^{-4}),$$

where k is an arbitrary function of $p\mathrm{T}^{-4}$.

The adiabatic law (2) is proved to be a consequence of this general law of absorption and of our fundamental hypothesis that the nature of the gas is constant on a level surface. It is also shown that radiation force and gravity (which are always opposite to each other and normal to the level surface) are in constant ratio within the partial body, when a law of the form (4) is valid.

Our results apply to all rotating systems of gaseous masses in radiative equilibrium, whether they consist of one or of several distinct bodies. They may also be shell- or ring-shaped, etc. The angular velocity and the density of the gas are immaterial.

In a recently published paper † of great interest, Eddington has studied the connection between the absolute magnitudes of stars and their masses. Eddington shows that the law of perfect gases is valid in the interior of dwarf stars also. His reason for this is that the extension of the atoms is greatly decreased by ionisation. It would thus be unnecessary any longer to base the study of radiative equilibrium in ordinary dwarf stars on v. d. Waals' equation. I have, however, retained v. d. Waals' equation, in order that the theory may also be used for the study of the interior of planets and exceptionally dense dwarf stars.

2. *Intensity of Radiation.*—Let us consider an arbitrary gaseous mass not necessarily in equilibrium. An arbitrary element of the gas round the point x, y, z has energy radiating through it in all directions. Let us consider specially a vector whose axis has direction-cosines α, β, γ, with the axis of x, y, z. Through an element of surface $d\mathrm{S}$

* *M.N.*, 84, 309 (1924). † *Ibid.*, 308 (1924).

normal to this vector, and within the angle of space $d\Sigma$, there passes per second the amount of energy

$$(5) \quad . \quad . \quad . \quad . \quad I(x, y, z; \alpha, \beta, \gamma) dS d\Sigma.$$

I is the intensity of radiation at the point x, y, z and in the direction α, β, γ.

The density of the gas ρ, its absolute temperature T, its coefficient of absorption k, and the intensity I are all functions of the length of path s, reckoned along the direction of the given vector.

Let us consider a small cylinder with base dS and height ds, whose axis is parallel to the vector. Through the base within a solid angle $d\Sigma$ passes an amount (5) of energy per second. Of this energy the quantity absorbed within the cylinder is

$$(6) \quad . \quad . \quad . \quad . \quad . \quad I dS d\Sigma . k\rho ds.$$

The cylinder, whose mass is $\rho dS ds$, emits within the angle of space $d\Sigma$, *i.e.* through the top of the cylinder, an energy per second

$$(7) \quad . \quad . \quad . \quad . \quad \rho dS ds . \mu T^4 . k . d\Sigma,$$

where μ is a physical constant

$$(8) \quad . \quad . \quad . \quad . \quad \mu = 1{\cdot}685 \cdot 10^{-5} \text{ (c, g, s)}.$$

Thus through the top of the cylinder passes a radiation per second of

$$(5) - (6) + (7).$$

But the radiation through the top of the cylinder is also

$$\left(I + \frac{dI}{ds} ds\right) dS d\Sigma.$$

By a comparison of the last two expressions we get the differential equation for the intensity of radiation:

$$(9) \quad . \quad . \quad . \quad . \quad \frac{dI}{ds} = -I . k\rho + \mu T^4 . k\rho.$$

This equation was first deduced by Schwarzschild.*

It is easy to integrate this equation. Introducing the so-called optical mass of the gas along the vector,

$$(10) \quad . \quad . \quad . \quad . \quad \sigma = \int_0^s k(s_1) . \rho(s_1) . ds_1$$

as a variable instead of the length of path s, we may write (9),

$$(11) \quad . \quad . \quad . \quad . \quad . \quad \frac{dI}{d\sigma} + I = \mu T^4.$$

* Göttingen, *Nachr. v. d. K. Ges. d. Wiss.*, 1906, p. 41 ; *Sitz-berichte d. K. preuss. Akad. d. Wiss.*, 28, 1914, p. 1183.

After integration we get for the intensity of radiation the integral expression

$$(12) \quad . \quad . \quad . \quad I(\sigma) = I_0 e^{-\sigma} + \int_0^{\sigma} e^{\sigma' - \sigma} \mu T^4(\sigma') d\sigma'.$$

The intensity of radiation inwards at the surface (for $\sigma = 0$, $s = 0$) is denoted by I_0. If the system consists of a single gaseous mass limited by a single convex surface, I_0 is $= 0$. But in other cases the intensity I_0 may be a function of the position of the point where the beam passes through the surface inwards as well as of the direction of the beam.

By successive partial integrations the following expression for $I(\sigma)$ may be obtained from (12):

$$I(\sigma) = \mu T^4(\sigma) - \frac{d\mu T^4(\sigma)}{d\sigma} + \frac{d^2\mu T^4(\sigma)}{d\sigma^2} - \int_0^{\sigma} e^{\sigma' - \sigma} \frac{d^3\mu T^4(\sigma')}{d\sigma'^3} d\sigma'$$
$$+ e^{-\sigma} \left\{ I_0 - \mu T^4(\sigma') + \frac{d\mu T^4(\sigma')}{d\sigma'} - \frac{d^2\mu T^4(\sigma')}{d\sigma'^2} \right\}_{\sigma'=0}.$$

Within the mass, and for rays leaving it, the optical mass σ is in general very great. For these cases the expression in the last line is practically zero. Within the mass, except for the nearest vicinity of its surface, the first three derivatives of T^4 are generally falling very rapidly. Thus for the interior the formula of three terms

$$(13) \quad . \quad . \quad . \quad . \quad I = \mu T^4 - \frac{d\mu T^4}{d\sigma} + \frac{d^2\mu T^4}{d\sigma^2}$$

is very approximately correct. It is independent of the ingoing radiation I_0. The formula (13) was first deduced by Milne.*

Finally (13) may be put into a form in which the dependence of the terms on x, y, z; α, β, γ, can be plainly seen. It follows from (10) that

$$(14) \quad . \quad . \quad . \quad \frac{d}{d\sigma} = \frac{d}{k\rho ds} = \frac{1}{k\rho} \left(\alpha \frac{d}{dx} + \beta \frac{d}{dy} + \gamma \frac{d}{dz} \right).$$

3. *The Equation of Radiative Equilibrium.*—The energy which a gaseous mass in radiative equilibrium loses in radiation is replaced from within by a continuous production of energy. Only a small part of this production can be ascribed to the secular contraction of the mass. It is thought that the energy is mainly created partly by the transformation of elements, *e.g.* by the conversion of hydrogen into helium and partly by transmutation of mass into energy.

The production of energy per second and gram within the star is denoted by $4\pi\epsilon$. P represents the emission, Q the absorption per second and gram. The equation of radiative equilibrium is evidently

$$(15) \quad . \quad . \quad . \quad . \quad . \quad 4\pi\epsilon = P - Q.$$

Now, according to the laws of Kirchhoff and Stefan,

$$(16) \quad . \quad . \quad . \quad . \quad . \quad P = 4\pi k \mu T^4.$$

* *M.N.*, 83, § 9, p. 122 (1923).

It is further clear that

$$(17) \quad . \quad . \quad . \quad . \quad . \quad Q = k\int I d\Sigma,$$

where $d\Sigma$ as before represents the surface element of the sphere of unit radius in the direction α, β, γ, and where the integration has to be carried out over the whole sphere. The expression (13) should be brought under the integral sign, taking into consideration the formula (14). We have

$$\int d\Sigma = 4\pi, \quad \int \alpha d\Sigma = 0, \ . \ . \ . \ ,$$

$$\int \beta\gamma d\Sigma = 0, \ . \ . \ . \ , \quad \int \alpha^2 d\Sigma = \frac{4\pi}{3}, \ . \ . \ .$$

Thus we get from (15), (16), and (17) the equation of radiative equilibrium

$$(18) \quad . \quad . \quad . \quad . \quad \epsilon = -\frac{1}{3\rho}\sum_{3} \frac{d}{dx}\left(\frac{d\mu T^4}{k\rho dx}\right).$$

The sum $\sum_{3}$ contains three terms corresponding to x, y, z. The equation (18) was given by Eddington * in the case of spherical symmetry and by Milne † in the general case.

4. *The Net Flow of Energy.*—Let us consider a cm.^{2} whose positive normal lies in the direction of a vector s'; the direction-cosines of s' are α', β', γ'. Through this cm.^{2}, within the solid angle $d\Sigma$ (with direction-cosines α, β, γ), flows an amount of energy per second

$$I . |\alpha\alpha' + \beta\beta' + \gamma\gamma'| . d\Sigma.$$

If this expression is integrated over the positive half of the sphere we get the amount of energy going towards the positive side. Integration over the negative half of the sphere gives us the quantity of energy going to the negative side. The net flow $F_{s'}$ in the direction of the vector s' is now defined as the difference between the amount of energy going towards the positive side and that going towards the negative side. Thus we get

$$F_{s'} = \int I . \sum_{3} \alpha\alpha' . d\Sigma,$$

where the integration is carried out over the whole surface of the sphere. After the expression (13) has been brought under the integral sign we get, with close approximation in the interior of the gaseous mass,

$$F_{s'} = -\frac{4\pi}{3k\rho}\sum_{3} \alpha' \frac{d\mu T^4}{dx},$$

i.e.

$$(19) \quad . \quad . \quad . \quad . \quad F_{s'} = -\frac{4\pi}{3k\rho}\frac{d\mu T^4}{ds'}.$$

* *Zeitschr. f. Physik*, 7, 356, 357 (1921).
† *M.N.*, 83, 123 (1923).

5. *Radiation-Force and Radiation-Pressure.*—Let us consider an element of unit mass through which passes a beam of radiation within the solid angle $d\Sigma$ and with direction-cosines α, β, γ. Of this radiation is absorbed an amount of energy per second

$$\delta Q = I k d\Sigma.$$

The electro-magnetic theory teaches that the element of mass is then affected by a force of radiation in the direction α, β, γ, of the size

$$\frac{I}{c}\delta Q,$$

where c is the velocity of light

(20) . . . . . $c = 3 \cdot 10^{10}$.

This theoretical result has been verified experimentally by Lebedew.

Our object is to calculate the projection $S_{s'}$ in the direction of a vector s' (with direction-cosines α', β', γ') of the total radiation-force which affects the element of unit mass. Clearly

$$S_{s'} = \frac{k}{c}\int I \cdot \sum_3 \alpha\alpha' \cdot d\Sigma,$$

i.e.

(21) . . . . . $S_{s'} = \frac{k}{c} F_{s'}$.

From (19) follows the approximate expression

(22) . . . . . $S_{s'} = -\frac{1}{\rho}\frac{d\frac{a}{3}T^4}{ds'}$

where

(23) . . . . . $a = \frac{4\pi\mu}{c} = 7{\cdot}060 \cdot 10^{-15}$.

Now the radiation-pressure ϖ is defined by means of the formula

(24) . . . . . $\rho S_{s'} = -\frac{d\varpi}{ds'}$.

We get therefore approximately

(25) . . . . . $\varpi = \frac{1}{3}aT^4$.

The formulæ (19), (22), (24), and (25) are quite general also when the gaseous mass is not in equilibrium.

6. *The Equation of Mechanical Equilibrium.*—We now assume that the gaseous mass is in mechanical equilibrium rotating about a fixed axis with constant angular velocity. The components of the gravity are then the partial derivatives of the gravity-potential Φ, which is the sum of the attractional potential and the centrifugal potential.

Let us consider a cylindrical element of height ds in the direction of

a vector s and with base $d\mathrm{S}$. The cylinder is affected by the force of gravity. Its projection along the axis of the cylinder is

$$\frac{d\Phi}{ds}\rho d\mathrm{S}ds.$$

The cylinder is also affected by the force of radiation. Its projection on the axis of the cylinder is

$$\mathrm{S}_s\rho d\mathrm{S}ds = -\frac{d\varpi}{ds}d\mathrm{S}ds.$$

On the surface of the cylinder the gas pressure p acts everywhere at right angles inwards. Only the pressure on the base and top of the cylinder gives rise to a component of force in the direction of the axis. This component is

$$-\frac{dp}{ds}ds d\mathrm{S}.$$

In mechanical equilibrium the sum of the three components mentioned is zero. Hence the equation of mechanical equilibrium is

$$(26) \quad . \quad . \quad . \quad . \quad d(p+\varpi) = \rho d\Phi.$$

The quantities p, ϖ, ρ, Φ are functions of the co-ordinates.

By a level surface is meant a surface along which

$$\Phi = \text{constant}.$$

It follows from (26) that $p+\varpi$ is constant on a level surface. Thus there exists a functional relation

$$(27) \quad . \quad . \quad . \quad . \quad p+\varpi = \mathrm{P}(\Phi).$$

The equation (26) then gives

$$\rho = \frac{d(p+\varpi)}{d\Phi} = \mathrm{P}'(\Phi).$$

Thus ρ is constant on a level surface.

According to v. d. Waals' formula the gas pressure is obtained by means of the relation

$$(28) \quad . \quad . \quad . \quad . \quad p = \frac{\mathsf{R}}{m}\frac{\rho}{1-b\rho}\mathrm{T}$$

where

$$(29) \quad . \quad . \quad . \quad . \quad \mathsf{R} = 8{\cdot}2962 \,.\, 10^7$$

is the gas-constant and m denotes the mean molecular weight of the gas mixture. The density of the gas at greatest compression is $1/b$.

Further the relation (25) is applicable within the mass.

The gas must be considered as a mixture of free electrons and more or less ionised atoms of different kinds. In calculating the mean

molecular weight the free electrons are included, and their mass may practically be neglected beside that of the atoms. In completely ionised hydrogen $m=\frac{1}{2}$; in completely ionised helium $m=\frac{4}{3}$. For other elements, on complete ionisation, m is in general near the value 2. Even when ionisation is fairly incomplete, m remains under 4. According to our fundamental hypothesis (p. 666) m and b are constant on a level surface.

The relations (27), (28), and (25) then show that T is constant on a level surface.

7. *The Production of Energy within the Gaseous Mass.*—I proceed to transform the second member of equation (18). In accordance with our fundamental hypothesis (p. 666) the coefficient of absorption k is constant on a level surface, *i.e.* k is a function of Φ.

Thus we have

$$\frac{1}{k\rho}\frac{d\mathrm{T}^4}{dx}=\frac{1}{k\rho}\frac{d\mathrm{T}^4}{d\Phi}\frac{d\Phi}{dx},$$

$$\frac{d}{dx}\left(\frac{1}{k\rho}\frac{d\mathrm{T}^4}{dx}\right)=\frac{1}{k\rho}\frac{d\mathrm{T}^4}{d\Phi}\frac{d^2\Phi}{dx^2}+\frac{d}{d\Phi}\left(\frac{1}{k\rho}\frac{d\mathrm{T}^4}{d\Phi}\right)\left(\frac{d\Phi}{dx}\right)^2.$$

The equation of radiative equilibrium (18) now becomes

$$(30)\quad \epsilon=-\frac{\mu}{3\rho}\left\{\frac{1}{k\rho}\frac{d\mathrm{T}^4}{d\Phi}\Delta\Phi+\frac{d}{d\Phi}\left(\frac{1}{k\rho}\frac{d\mathrm{T}^4}{d\Phi}\right)\left(\frac{d\Phi}{dn}\right)^2\right\},$$

since

$$\left(\frac{d\Phi}{dn}\right)^2=\sum_3\left(\frac{d\Phi}{dx}\right)^2.$$

From Poisson's equation it follows that

$$(31)\quad \Delta\Phi=-4\pi\mathrm{G}\rho+2\omega^2.$$

It will be shown later, in § 11, that

(32) . . . . . $\dfrac{d\Phi}{dn}$ varies on the level surface within a *rotating* body in mechanical equilibrium.

The right member of (30) consists of two terms. The first is constant on the level surface. The second is variable, when it is not zero. In accordance with our fundamental hypothesis (p. 666) ϵ is constant on a level surface. Thus the second term in (30) must vanish, and our fundamental hypothesis leads to the consequence

$$\frac{d}{d\Phi}\left(\frac{1}{k\rho}\frac{d\mathrm{T}^4}{d\Phi}\right)=0,$$

i.e. to the relation

$$(33)\quad \rho d\Phi=\mathrm{C}\frac{a}{3}\frac{d\mathrm{T}^4}{k},$$

where C is constant within each partial body of the system.

By means of (30), (31), (33), and (23), we get the expression

$$(34) \quad . \quad . \quad . \quad . \quad \epsilon = \frac{cG}{C}\left(1 - \frac{\omega^2}{2\pi G\rho}\right).$$

Our general fundamental hypothesis thus leads to a very special relation between ϵ and ρ within the gaseous mass and also to the simpler equation (33) of radiative equilibrium.

In deducing the law (34) no special assumption has been made as to the quantities m, b, and k. They may be entirely arbitrary functions of the composition and state of the gas. The formula (34) is correct even if the laws of Kirchhoff, Stefan, Boyle, and v. d. Waals were not applicable within the stars.

The law (34) is reduced, if $\omega^2 = 0$, to Eddington's hypothesis—that $\epsilon =$ constant within the gaseous mass. By passing from the general case ($\omega^2 \neq 0$) to the special one ($\omega^2 = 0$) we have succeeded in finding firmer grounds for Eddington's hypothesis than was previously possible. It is to be noted that Eddington's hypothesis does not follow from our fundamental hypothesis if we are only dealing with a star without rotation. For in that case $\frac{d\Phi}{dn}$ is constant on every level surface. We cannot then proceed to the relation (33).

I will now show, assuming only that m, k, and b are constant over the level surface, *that ϵ cannot be constant within a rotating gaseous mass in radiative equilibrium unless the density is constant also.*

For let us suppose that ϵ is constant and ρ variable within the rotating gaseous mass. Then a relation of the form (34) cannot exist; and therefore a relation of the form (33) cannot exist either. It then follows from (30) and from (32), which will be proved later, that ϵ is not constant even along a level surface.

Thus Milne's hypothesis, according to which ϵ is constant within a rotating star, cannot be correct for non-homogeneous stars of constant nature along level surfaces. For such rotating gaseous masses the law (34) holds good.

The generation of energy is positive within the region where $\rho > \frac{\omega^2}{2\pi G}$. Close to the surface the formula (34) perhaps ceases to apply. In an intermediate region, where $\rho < \frac{\omega^2}{2\pi G}$, the production of energy is negative. This implies that radiative energy is transformed into potential energy or into matter. This region reaches a good way inwards even for nearly spherical stars. In a following paper it will be shown that the depth of this region with the radius as unity, is $= \cdot 0903\left(\frac{\omega^2}{2\pi G\rho_m}\right)^{1/3} = \cdot 0985$ (oblateness)$^{1/3}$. *E.g.* for the sun, considered as a perfect gas star with constant rotation, the density has the value $\left(\frac{\omega^2}{2\pi G}\right)$ at a depth of 2 per cent. of the radius.

8. *Net Flux, Surface Brightness, Effective Temperature, and Total Radiation.*—With net flux is meant the net flow of energy per second

and cm.^{2} through the level surface. In accordance with (19), (23), and (33) the net flux has the expression

$$(35) \quad . \quad . \quad . \quad F = F_n = -\frac{4\pi\mu}{3k\rho}\frac{dT^4}{d\Phi}\frac{d\Phi}{dn} = \frac{c}{C}\frac{d\Phi}{dn_i}.$$

Here dn_i is positive in the direction of the inner normal of the level surface.

The formulæ collected or deduced above are very nearly true for the interior of the gaseous mass, but not too close to its limiting surface. We must assume that the production of energy in the neighbourhood of the surface is relatively insignificant compared with the total radiation. The net flux close beneath the limiting surface gives thus a good approximation to the surface brightness, *i.e.* the net flux through the limiting surface. Again the formula (35) applied to the limiting surface

$$(36) \quad . \quad . \quad . \quad . \quad . \quad \tilde{F} = \frac{c}{C}\frac{d\tilde{\Phi}}{dn_i}$$

gives, in consequence of the continuity of the gravity, an approximate expression for the net flux close under the limiting surface. Thus (36) gives an approximately correct expression of the surface brightness.

The outer layers of gas tend by their absorption, emission, and diffusion to deprive the outgoing radiation to some extent of the character of black body radiation, which it actually possesses at greater depths below the surface. Nevertheless, as a first approximation, the outgoing radiation may be said to follow Stefan's law.

T_e represents the effective temperature, *i.e.* the temperature of the black body with the surface brightness $\tilde{F}$. Then

$$(37) \quad . \quad . \quad . \quad T_e^4 = \frac{\tilde{F}}{\pi\mu} = \frac{4}{ac}\tilde{F} = \frac{4}{aC}\frac{d\tilde{\Phi}}{dn_i}.$$

We have not yet made any assumption as to the number of partial bodies composing the rotating system, nor of the density of the gas, nor of the angular velocity. We have only supposed that the nature of the gas is constant over the level surface. Our results have therefore a very wide extent. They apply to ordinary giant or dwarf stars. But they also apply to transition forms between single and double stars, to double and multiple stars, and to shell- and ring-formed figures which a rotating gaseous mass in radiative equilibrium can take.

The level surfaces along which Φ is constant may be divided into ordinary and singular level surfaces. On the former $\frac{d\Phi}{dn} \neq 0$; on the latter $\frac{d\Phi}{dn} = 0$, at least along certain curves or at single points. An ordinary level surface has throughout a definite tangent plane. On a singular level surface, on the other hand, edges or points appear at which $\frac{d\Phi}{dn} = 0$. Singular level surfaces are transition forms, in which the general character of the ordinary surfaces is changing.

Let us, for example, consider a singular level surface in which a series of closed surfaces is undergoing transition into other series of open surfaces. In a state of equilibrium the gas cannot then extend through the singular surface—in other words, the density must be zero, either before or at latest on the singular surface. Otherwise the gas would escape through an opening.

For another example we choose the case in which two series of closed level surfaces, each including its maximum value of Φ, coalesce in a single series of closed level surfaces, which are then at first hour-glass-shaped. The singular surface is then in one place pinched into a double point. If the gas extends past the singular surface the star has two nuclei surrounded by a common gaseous envelope. But if the gas does not reach the pinched singular surface the star is a double star.

If the limiting surface is singular the edges or points must appear dark. This is perhaps the explanation of the dark equatorial belt of certain lenticular nebulæ.

Let us return to the formula (34). After integration of $4\pi\epsilon\rho dxdydz$ over the partial body we get for its total radiation the expression

$$(38) \quad . \quad . \quad . \qquad L = \frac{4\pi cG}{C} M\left(1 - \frac{\omega^2}{2\pi G\rho_m}\right),$$

in which the mean density of the partial body is denoted by ρ_m and its mass by M.

Poincaré,* neglecting radiation-force, has shown that

$$\frac{\omega^2}{2\pi G\rho_m} < 1,$$

for otherwise gravity would at some place on the surface be directed outwards. Evidently this formula holds good also when radiation-force is taken into consideration, because $L > 0$.

9. *Relation between Gas Pressure and Temperature. Different Assumptions as to the Coefficient of Absorption.*—In accordance with (26) and (33) and taking into account (25) we obtain the following differential equation between p and T,

$$(39) \quad . \quad . \quad . \quad . \qquad dp = \frac{a}{3}\left\{\frac{C}{k} - 1\right\} dT^4.$$

We may regard k as a function of p and T. Different assumptions concerning this function give rise to different theories of radiative equilibrium.

Eddington has successively proposed several hypotheses. At first he supposed by way of trial that †

$$(40) \quad . \quad . \quad . \quad . \quad . \quad k = \text{constant}.$$

With the notation

$$(41) \quad . \quad . \quad . \quad . \quad . \quad C = \frac{k}{1-\beta},$$

* *Figures d'equilibre d'une masse fluide*, Paris, 1902, p. 11.
† *M.N.*, 77, 24 (1916).

where β is a constant between 0 and + 1, he arrived at the functional connection

$$(42) \quad . \quad . \quad . \quad . \quad p = \frac{a}{3}\frac{\beta}{1-\beta}T^4.$$

This particular solution of (39) is characterised by the condition that $T = 0$ on the surface. The general solution, where $T = \tilde{T}$ for $p = 0$, leads through (33) to the absurd consequence that $\Phi = \infty$ when $T = \tilde{T}$, *i.e.* on the limiting surface.

Later Eddington * supposed more in accordance with modern emission theories that $k \propto \rho m^{-1} T^{-3}$ for a perfect gas. We are writing this hypothesis in the form

$$(43) \quad . \quad . \quad . \quad . \quad . \quad k = K p T^{-4},$$

where K is a constant. In (39) we introduce instead of C a constant β by substituting

$$(44) \quad . \quad . \quad . \quad . \quad C = \frac{a}{3} K \frac{\beta}{(1-\beta)^2}.$$

Even by the hypothesis (43) the particular solution (42) holds good. The general solution, where $T \neq 0$ for $p = 0$, can easily be formed; it is to be rejected because Φ must be finite on the surface. Evidently the relation (41) is still to be applied as in the first hypothesis.

From the relations (25) and (42) it follows that the gas-pressure and the radiation pressure are in constant relation to each other within the partial body.

V. d. Waals' equation (28) compared with (42) gives the relation

$$(45) \quad . \quad . \quad . \quad . \quad \frac{R}{m}\frac{\rho}{1-b\rho} = \frac{a}{3}\frac{\beta}{1-\beta}T^3,$$

which may also be written

$$(45^1) \quad . \quad . \quad . \quad . \quad \frac{1}{\rho} = b + \frac{R}{m}\frac{3}{a}\frac{1-\beta}{\beta}T^{-3}.$$

From (33), after integration from the limiting surface, where $\Phi = \tilde{\Phi}$, $T = 0$, we get

$$(46) \quad . \quad . \quad . \quad \Phi - \tilde{\Phi} = \frac{a}{3}\frac{b}{1-\beta}T^4 + \frac{R}{m}\frac{4}{\beta}T.$$

If T is eliminated between (45) and (46) we find a relation between ρ and Φ of the form

$$(47) \quad . \quad . \quad b\rho\frac{(1-\frac{3}{4}b\rho)^3}{(1-b\rho)^4} = \frac{ab}{192}\left(\frac{m}{R}\right)^4\frac{\beta^4}{1-\beta}(\Phi-\tilde{\Phi})^3.$$

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* *Zeitschr. f. Physik*, **7**, 389 (1921); *M.N.*, **84**, 309 (1924).

The above formulæ are simplified if $b\rho$ can be neglected in comparison with unity. Thus for a perfect gas

$$(48) \quad \rho = \frac{m}{R}\frac{a}{3}\frac{\beta}{1-\beta}T^3,$$

$$(49) \quad \Phi - \tilde{\Phi} = \frac{R}{m}\frac{4}{\beta}T,$$

$$(50) \quad \rho = \frac{a}{192}\left(\frac{m}{R}\right)^4 \frac{\beta^4}{1-\beta}(\Phi - \tilde{\Phi})^3.$$

The expression (38) for the total radiation of the partial body takes different forms in the two cases (40) and (43). In the former case

$$(51) \quad L = \frac{4\pi cG}{k}(1-\beta)M\left(1 - \frac{\omega^2}{2\pi G\rho_m}\right),$$

but in the second case we have

$$(52) \quad L = \frac{12\pi cG}{aK}\frac{(1-\beta)^2}{\beta}M\left(1 - \frac{\omega^2}{2\pi G\rho_m}\right).$$

The constant β has in either case the same meaning

$$(53) \quad \beta = \frac{p}{p+\varpi}.$$

But β has also another meaning. Radiation force, whose direction coincides with the outer normal of the level surface, has, according to (24), (25), and (33), the expression

$$(54) \quad S_n = -\frac{1}{\rho}\frac{d\varpi}{dn} = -\frac{1}{\rho}\frac{a}{3}\frac{dT^4}{d\Phi}\frac{d\Phi}{dn} = -\frac{k}{C}\frac{d\Phi}{dn}.$$

Thus the radiation force is a fractional part of the gravity and acts in opposite direction. In the two cases (40) and (43) the formula (54), with regard to (41), reduces to

$$(54^1) \quad S_n = -(1-\beta)\frac{d\Phi}{dn}.$$

Hence $1-\beta$ is the constant ratio between radiation force and gravity.

Evidently the two assumptions (40) and (43) are special cases of a more general formula

$$(55) \quad k = k(p T^{-4}),$$

where $k(x)$ is an arbitrary function. Introducing instead of C the constant β by the substitution

$$(56) \quad C = \frac{1}{1-\beta}k\left(\frac{a}{3}\frac{\beta}{1-\beta}\right)$$

we find again the relations (42), (41), (45), . . . , (50). But the total radiation then becomes because of (38) and (56)

$$(57) \quad . \quad . \quad L = 4\pi cG \frac{1-\beta}{k\left(\frac{a}{3}\frac{\beta}{1-\beta}\right)} . M . \left(1 - \frac{\omega^2}{2\pi G\rho_m}\right).$$

The constant β has the same meaning as before.

10. *Determination of the Gravity-Potential.*—Assuming that the coefficient of absorption k is a known function of the gas pressure p and the temperature T we get, after integrating (39), the relation between p and T. Further, the gravity-potential Φ is fixed as a function of the temperature after (33) has been integrated. Thus ρ, T and p can be expressed as functions of Φ.

If the coefficient of absorption k follows the law (40) or the law (43) or even the more general law (55) the functional relation between ρ and Φ is of the form (47) or (50) respectively.

It remains lastly to determine the gravity-potential Φ as a function of the co-ordinates. Φ is the sum of the attractional and centrifugal potentials. The latter is of the form $\frac{\omega^2}{2}s^2$, where s is the distance from the axis of rotation. Thus from the equations of Poisson and Laplace, Φ satisfies the differential equation

$$(58) \quad . \quad . \quad . \quad . \quad \Delta\Phi + 4\pi G\rho - 2\omega^2 = 0,$$

where

$$\Delta = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

is Laplace's differential symbol. The system may consist of several partial bodies rotating together as a rigid body about a fixed axis. Outside the partial bodies $\rho = 0$. Inside them, on the other hand, (47) or (50) are valid, the latter in the case of perfect gases. Thus we get as parameters in the first place the angular velocity ω, and in addition different values of β and $\tilde{\Phi}$ for every partial body. The differential equations of the problem thus contain $2n+1$ parameters, if the number of partial bodies is n. The required solution must be such that the function $\Phi - \frac{\omega^2}{2}s^2$ possesses the characteristic properties of the attraction potentials: this function and its derivatives of the first order must be finite and continuous through the whole of space, even at the surfaces of the partial bodies.

At present it would appear that the problem can only be solved in cases when the partial bodies are nearly spherical.

In two later papers I propose, taking as starting-point the functional relation (47), to deal with equilibrium first in a slightly oblate rotating single star or planet, and secondly for a double star system with bound rotation, in which the distance between the components is great in proportion to the mean radii of the stars. It will be seen that the

gravity-potential depends in the first case ($n=1$) on three parameters, and in the second ($n=2$) on five. As fundamental parameters we may choose the masses, the mean densities, and the angular velocity of the system. The parameters β and $\tilde{\Phi}$ can be expressed as functions of these fundamental parameters. The form and extent of the system and its radiation are determined in all respects by the fundamental parameters.

11. *On the Non-existence of Rotating Equilibrium Figures with Parallel Level Surfaces.*—I will now proceed to show that the level surfaces of a rotating liquid or gaseous mass in mechanical equilibrium cannot be parallel.

For suppose that a series of such equilibrium figures exists for arbitrary values of the angular velocity ω. Let us first consider small values of ω, and let us assume that the functions and constants that appear can be expanded in series of ascending powers of ω^2.

By Φ and Ψ are denoted the inner and outer gravity-potentials of the mass. Between Φ and the density ρ there is a certain functional relation

$$-4\pi G\rho = f(\Phi). \tag{59}$$

The independent variables are the radius-vector r, $u=\frac{z}{r}$, and the longitude v. Φ and Ψ satisfy the differential equations

$$\Delta\Phi + 4\pi G\rho - 2\omega^2 = 0, \quad \Delta\Psi - 2\omega^2 = 0, \tag{60}$$

where

$$\Delta = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d}{du}\left[(1-u^2)\frac{d}{du}\right] + \frac{1}{r^2(1-u^2)}\frac{d^2}{dv^2}$$

is the symbol of Laplace in polar co-ordinates. The functions Φ and Ψ possess the characteristic properties of the gravity-potential. Thus Φ is finite inside, and $\Psi - \frac{\omega^2}{2}r^2(1-u^2)$ is finite outside the surface of the body. At this surface whose equation is

$$\Phi = C \tag{61}$$

the conditions of continuity

$$\Phi - \Psi = 0, \quad \dot{\Phi} - \dot{\Psi} = 0 \text{ (when } \Phi = C) \tag{62}$$

hold good. (Derivation with regard to r is indicated by a point.)

The parallel condition may be formulated

$$\frac{d\Phi}{dn} = P(\Phi), \tag{63}$$

for if $\frac{d\Phi}{dn}$ is constant along the level surface, then $\frac{d\Phi}{dn}$ is a function of Φ.

The above functions are expanded according to powers of ω^2 in the form

$$(64)\quad \begin{cases} \Phi = \Phi_0 + \Phi_1 + \ldots, \\ \Psi = \Psi_0 + \Psi_1 + \ldots, \\ f(\Phi) = f_0(\Phi) + f_1(\Phi) + \ldots, \\ \mathrm{P}(\Phi) = \mathrm{P}_0(\Phi) + \mathrm{P}_1(\Phi) + \ldots, \\ \mathrm{C} = \mathrm{C}_0 + \mathrm{C}_1 + \ldots. \end{cases}$$

where the index shows the degree of the term in regard to ω^2.

Φ_0 and Ψ_0 are functions of r alone satisfying the ordinary differential equations

$$(65)\quad \ddot{\Phi}_0 + \frac{2}{r}\dot{\Phi}_0 = f_0(\Phi_0), \quad \ddot{\Psi}_0 + \frac{2}{r}\dot{\Psi}_0 = 0.$$

We have

$$(66)\quad \Psi_0 = \frac{\mathrm{GM}}{r}$$

where M is the mass of the body.

The radius of the spherical equilibrium figure is denoted by $\bar{r}$. A straight line over a function signifies the value or expression of the function for $r = \bar{r}$.

We have, because of (59), $f_0(\bar{\Phi}_0) = 0$ for gases and $f_0(\bar{\Phi}_0) \neq 0$ for liquids.

According to (62)

$$(67)\quad \bar{\Phi}_0 - \bar{\Psi}_0 = 0, \quad \bar{\dot{\Phi}}_0 - \bar{\dot{\Psi}}_0 = 0,$$

and in accordance with (65) we also get

$$(68)\quad \bar{\ddot{\Phi}}_0 - \bar{\ddot{\Psi}}_0 = f_0(\bar{\Phi}_0).$$

The functions Φ_1 and Ψ_1 satisfy the differential equations

$$(69)\quad \Delta\Phi_1 = 2\omega^2 + f'_0(\Phi_0)\cdot\Phi_1 + f_1(\Phi_0), \quad \Delta\Psi_1 = 2\omega^2.$$

(Derivation with regard to ϕ is indicated by an accent.)

From the conditions (62) we can derive certain conditions which Φ_1 and Ψ_1 must satisfy when $r = \bar{r}$. The equation (61) of the limiting surface may be written

$$\Phi_0 + \Phi_1 + \ldots = \mathrm{C}_0 + \mathrm{C}_1 + \ldots.$$

We have

$$\mathrm{C}_0 = \bar{\Phi}_0 = \bar{\Psi}_0 = \frac{\mathrm{GM}}{\bar{r}}.$$

From the equation of the limiting surface we get, after solution

$$(70)\quad r - \bar{r} = \frac{\bar{\Phi}_1 - \mathrm{C}_1}{-\bar{\dot{\Phi}}_0} = \frac{\bar{r}^2}{\mathrm{GM}}(\bar{\Phi}_1 - \mathrm{C}_1).$$

The conditions (62) are expanded according to powers of $r-\bar{r}$. Introducing (70) we get, as a consequence of (67_2) and (68), if terms in ω^4, ω^6, . . . are omitted, the exact conditions

$$(71) \quad . \quad . \quad \bar{\Phi}_1-\bar{\Psi}_1=0, \quad \dot{\bar{\Phi}}_1-\dot{\bar{\Psi}}_1=-f_0(\bar{\Phi}_0)\frac{\bar{r}^2}{GM}(\bar{\Phi}_1-C_1).$$

These are the looked for conditions for Φ_1 and Ψ_1.

I pass to the parallel condition (63). If ω^4 is neglected, then

$$\frac{d\Phi}{dn}=\dot{\Phi}+\ldots.$$

With the same approximation (63) can therefore be written

$$\dot{\Phi}_0+\dot{\Phi}_1+\ldots=P_0(\Phi_0)+P'_0(\Phi_0)\Phi_1+P_1(\Phi_0)+\ldots.$$

This equation splits up into the following exact relations

$$\begin{aligned}\dot{\Phi}_0&=P_0(\Phi_0),\\ \dot{\Phi}_1&=P'_0(\Phi_0)\Phi_1+P_1(\Phi_0),\\ &\ldots\ldots\ldots\end{aligned}$$

From the first, after derivation with regard to r, we get

$$\ddot{\Phi}_0=P'_0(\Phi_0)\dot{\Phi}_0.$$

Using this, we get from the second, when divided by $\dot{\Phi}_0$,

$$\frac{d}{dr}\left(\frac{\Phi_1}{\dot{\Phi}_0}\right)=\frac{P_1(\Phi_0)}{\dot{\Phi}_0}.$$

After integration we get

$$(72) \quad . \quad . \quad . \qquad \Phi_1=\chi_1(r)+\dot{\Phi}_0(r)\,.\,g(u,v)$$

where

$$\chi_1(r)=\dot{\Phi}_0\int\frac{P_1(\Phi_0)}{\dot{\Phi}_0}dr$$

is a function of r alone, while $g(u, v)$ is a function of u and v, independent of r, and at present unknown.

The function $g(u, v)$ can be expanded into spherical harmonics. Conserving only the spherical harmonic $X_2(u)=\frac{3u^2-1}{2}$, we get

$$\Phi_1=A\dot{\Phi}_0X_2(u)+\ldots$$

where A is a constant.

This expression for Φ_1 is now introduced in the first equation (69). According to the theory of spherical harmonics we have

$$\Delta(\dot{\Phi}_0X_2)=\left(\dddot{\Phi}_0+\frac{2}{r}\ddot{\Phi}_0-\frac{6}{r^2}\dot{\Phi}_0\right)X_2.$$

Further, the first of the equations (65) gives, after derivation,

$$\ddot{\Phi}_0+\frac{2}{r}\dot{\Phi}_0-\frac{2}{r^2}\Phi_0=f'_0(\Phi_0)\dot{\Phi}_0.$$

Thus

$$\Delta\Phi_1=\left[f_0(\Phi_0)-\frac{4}{r^2}\right]\Phi_0 \mathrm{A}X_2(u)+\ldots.$$

Now $X_2(u)$ (as well as all the other spherical harmonics) must disappear out of (69_1). Hence it follows that

$$-\frac{4}{r^2}\Phi_0 \mathrm{A}=0,$$

i.e.

$$\mathrm{A}=0.$$

Legendre's function $X_2(u)$ is thus lacking in the expansion of Φ_1 into spherical harmonics.

On the other hand, the function

$$\Psi_1-\frac{\omega^2}{3}r^2[1-X_2(u)]$$

is finite $(=0)$ for $r=\infty$. Thus, regarding (69_2), the coefficient of $X_2(u)$ in the expansion of Ψ_1 in spherical harmonics is of the form

$$\left(-\frac{r^2}{3}+\frac{\mathrm{B}}{r^3}\right)\omega^2,$$

where B is a constant.

When $r=\bar{r}$ Φ_1 and Ψ_1 must satisfy the conditions (71) for all values of u and v. The different spherical harmonics must all disappear out of these equations. If $X_2(u)$ is to disappear the constant B must satisfy the two conditions

$$\left(\frac{\bar{r}^2}{3}-\frac{\mathrm{B}}{\bar{r}^3}\right)\omega^2=0,$$

$$\left(\frac{2\bar{r}}{3}+\frac{3\mathrm{B}}{\bar{r}^4}\right)\omega^2=0.$$

They are inconsistent if $\omega^2\neq 0$.

Hence it follows that the gravity-potential of a slowly rotating liquid or gaseous mass in mechanical equilibrium cannot fulfil the parallel condition.

We now let the angular velocity assume higher and higher values. It is clear that the level surfaces must remain non-parallel, at least so long as the character of the equilibrium does not change essentially, *e.g.* by passing through some bifurcation form in which new forms of a different character attach themselves.

In this way the theorem (32) is proved under fairly general postulates.

The Radiative Equilibrium of a Slightly Oblate Rotating Star.

By H. v. Zeipel, Assoc. R.A.S.

1. *Introduction and Summary.*—In the previous paper (p. 665), here denoted by I, the general features of radiative equilibrium in a rotating system of gaseous masses were studied. As a fundamental hypothesis we assumed that the nature (composition) of the gas is constant along each level surface. From the further assumption that the absorption-coefficient is a function of the product pT^{-4}, where p is the gas pressure and T the temperature, a simple relation between the density and the gravity-potential was obtained [I, (47), (50)]. In that paper, as in this one, it was assumed that the gas follows v. d. Waals' equation. True, Eddington * has recently shown that the more special law of perfect gases is generally applicable, even to dwarf stars, since the size of the atoms is reduced in a high degree by ionisation. But gaseous planets, such as Jupiter, Saturn, and perhaps also the Earth (in its interior), have a relatively low temperature, and the ionisation is consequently incomplete. In the case of planets and certain dwarf stars of very great density it would therefore seem necessary to have recourse to v. d. Waals' equation.

If the system has a single boundary surface enclosing the centre of gravity the *form* of its equilibrium figure depends in general on two parameters, mean density ρ_m and angular velocity ω. But in the case of a perfect gas, the form is determined by the single parameter $\omega^2/2\pi G\rho_m$, where G is the attraction constant. It has not yet been possible to make a complete mathematical study of these equilibrium forms and their evolution series. A very important group of equilibrium forms can, however, even now be treated in detail with the help of the spherical harmonics. The characteristic of these equilibria is that the oblateness is slight.

In this paper the radiative equilibrium of a slightly oblate star of arbitrary central density is studied. The unknown quantities are expanded according to powers of the oblateness. The first two terms of the series are derived. The relation between mean density, angular velocity, and oblateness is deduced. The variation of gravity, surface brightness, and effective temperature with latitude is studied. For perfectly gaseous stars the theory is also worked out numerically.

2. *The Differential Equations of the Problem.*—Φ and Ψ represent the internal and external gravity-potentials respectively, *i.e.* the sum of the centrifugal potential and the internal or external attraction potential of the body. The differential equations

$$(1) \quad . \quad . \quad \Delta\Phi + 4\pi G\rho - 2\omega^2 = 0, \quad \Delta\Psi - 2\omega^2 = 0,$$

follow directly from the equations of Poisson and Laplace.

Between the density ρ of the gas and the gravity-potential Φ exists a relation of the form [I, (47)]

$$(2) \quad . \quad . \quad b\rho\frac{(1-\frac{3}{4}b\rho)^3}{(1-b\rho)^4} = \frac{ab}{3\cdot 4^3}\left(\frac{m}{R}\right)^4\frac{\beta^4}{1-\beta}(\Phi-\tilde{\Phi})^3.$$

* *Monthly Notices of R.A.S.*, **84**, 308.

Here $1/b$ signifies the greatest possible density of the gas according to v. d. Waals' formula; m is the mean molecular weight of the gas, β is the constant ratio of gas pressure and total pressure within the star, $\tilde{\Phi}$ is the constant value of Φ on the boundary surface; the constants a and R have the following values

$$a = 7{\cdot}060 \,.\, 10^{-15},$$
$$\mathsf{R} = 8{\cdot}2962 \,.\, 10^{7}.$$

For a perfect gas the constant b is zero and the formula (2) reduces to

$$(3) \quad \rho = \frac{a}{3 \cdot 4^3}\left(\frac{m}{\mathsf{R}}\right)^4 \frac{\beta^4}{1-\beta}(\Phi - \tilde{\Phi})^3.$$

The constants β, $\tilde{\Phi}$, and ω are the original parameters of the problem. By the linear transformation

$$(4) \quad \Phi - \tilde{\Phi} = \mathrm{G}\frac{h}{l}\phi, \quad \Psi - \tilde{\Phi} = \mathrm{G}\frac{h}{l}\psi,$$

the scalar potentials ϕ and ψ are introduced. The constant factor $\mathrm{G}\frac{h}{l}$ is chosen so that

(5) . . $\phi = 1$ at the centre of gravity of the star.

We have

(6) . . . $\phi = 0$ at the boundary surface.

Instead of the radius vector r, reckoned from the centre of gravity, ξ is introduced by the relation

$$(7) \quad r = l\xi,$$

where l is a certain length which will shortly be fixed. Other independent variables are u, which is the cosine of the angle between the radius-vector and the axis of rotation, and the longitude v.

Through the formula

$$(8) \quad \kappa = \frac{ab}{3 \cdot 4^3}\left(\frac{m}{\mathsf{R}}\right)^4 \frac{\beta^4}{1-\beta}\left(\mathrm{G}\frac{h}{l}\right)^3$$

a parameter κ is defined.

Instead of the density ρ we introduce

$$(9) \quad \chi = \frac{b}{\kappa}\rho.$$

Between χ and ϕ exists, because of (2), a relation of the form

$$(10) \quad \frac{\chi(1 - \frac{3}{4}\kappa\chi)^3}{(1-\kappa\chi)^4} = \phi^3.$$

If the length l mentioned above is fixed by the condition

$$(11) \quad . \quad . \quad . \quad . \quad 4\pi h^{-1} l^3 . \kappa b^{-1} = 1,$$

and if the new parameter λ is defined by the notation

$$(12) \quad . \quad . \quad . \quad \lambda = G^{-1} h^{-1} l^3 . 2\omega^2 = \frac{\omega^2 b}{2\pi G \kappa},$$

then the equations (1) of Poisson and Laplace appear in the forms

$$(13) \quad . \quad . \quad . \quad \nabla\phi + \chi - \lambda = 0, \quad \nabla\psi - \lambda = 0.$$

Here ∇ is Laplace's differential symbol in scalar polar co-ordinates,

$$\nabla = \frac{d^2}{d\xi^2} + \frac{2}{\xi}\frac{d}{d\xi} + \frac{1}{\xi^2}\frac{d}{du}\left[(1-u^2)\frac{d}{du}\right] + \frac{1}{\xi^2(1-u^2)}\frac{d^2}{dv^2}.$$

The scalar potentials ϕ and ψ satisfy the following conditions

$$(14) \quad . \quad \phi - \psi = 0, \quad \dot{\phi} - \dot{\psi} = 0 \text{ at the boundary surface ;}$$

$$(15) \quad . \quad . \quad . \quad . \quad \phi = 1, \quad \text{when } \xi = 0.$$

$$(16) \quad . \quad \psi = \frac{\lambda}{4}\xi^2(1-u^2) - \gamma + \frac{\delta}{\xi} + \frac{1}{\xi^3} P\left(\frac{1}{\xi}\right) \text{ for great values of } \xi.$$

The point over a letter means derivation with respect to ξ. The quantities γ and δ are constants which, as will be shown later, depend on κ and λ. Terms in $\frac{1}{\xi^2}$ are lacking in (16), because the origin is chosen at the centre of gravity. $P\left(\frac{1}{\xi}\right)$ is a series of ascending powers of $\frac{1}{\xi}$.

The scalar differential equations (13) contain only two parameters κ and λ. For perfectly gaseous stars $\kappa = 0$.

3. *Expansions according to Powers of* λ.—Assume now that the parameter λ is a small quantity. The unknown functions and constants are developed in series according to powers of λ under the form

$$(17) \quad . \quad . \quad . \quad . \quad \begin{cases} \phi = \phi_0 + \lambda\phi_1 + \ldots, \\ \psi = \psi_0 + \lambda\psi_1 + \ldots, \\ \chi = \chi_0 + \lambda\chi_1 + \ldots, \\ \gamma = \gamma_0 + \lambda\gamma_1 + \ldots, \\ \delta = \delta_0 + \lambda\delta_1 + \ldots. \end{cases}$$

From (10) we have

$$(18) \quad . \quad . \quad . \quad . \quad \frac{\chi_0(1 - \frac{3}{4}\kappa\chi_0)^3}{(1-\kappa\chi_0)^4} = \phi_0{}^3.$$

Further

$$(19) \quad . \quad . \quad . \quad . \quad . \quad \chi_1 = E . \phi_1,$$

where

$$(20) \quad . \quad . \quad . \qquad E = \frac{d\chi_0}{d\phi_0} = 3\chi_0^{2/3}(1 - \kappa\chi_0)^{7/3}.$$

The differential equations (13) now split up into the following pairs of equations

$$(21) \quad . \quad . \quad . \qquad \nabla\phi_0 + \chi_0 = 0, \qquad \nabla\psi_0 = 0;$$

$$(22) \quad . \quad . \quad . \qquad \nabla\phi_1 + \chi_1 - 1 = 0, \qquad \nabla\psi_1 - 1 = 0;$$

$$\nabla\phi_2 + \chi_2 = 0, \qquad \nabla\psi_2 = 0;$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

In order to express the conditions which may be deduced from (14) we must study the equation (6) of the boundary surface more closely. Here and subsequently the terms in λ^2 will not be written out, though they can be obtained without difficulty. The equation of the boundary surface is

$$(23) \quad . \quad . \quad . \quad . \qquad \phi_0 + \lambda\phi_1 + \ldots = 0.$$

I will assume that ϕ_0 (and ψ_0) depend only on ξ, *i.e.* that they are independent of u and v. This implies that at vanishing angular velocity the body has spherical symmetry. We have $\phi_0 = 1$, $\dot{\phi}_0 = 0$, when $\xi = 0$ [*cf.* (35)]; this defines the function ϕ_0 fully. The first zero of ϕ_0 is denoted by $\bar{\xi}$. $\bar{\xi}$ is a function of κ. It will be seen that ϕ_0 and ψ_0 are regular when $\xi = \bar{\xi}$. A line over a function indicates the value or expression of the function when $\xi = \bar{\xi}$.

Now at the boundary surface (23)

$$\bar{\dot{\phi}}_0(\xi - \bar{\xi}) + \lambda\bar{\phi}_1 + \ldots = 0.$$

After solution of this equation we get ξ at the boundary surface as a function of u and v in the form of a series

$$(24) \quad . \quad . \quad . \qquad \xi = \bar{\xi} + \lambda\xi_1 + \lambda^2\xi_2 + \ldots$$

The coefficient of λ in this expansion is clearly

$$(25) \quad . \quad . \quad . \quad . \quad . \qquad \xi_1 = \frac{\bar{\phi}_1}{-\bar{\dot{\phi}}_0}.$$

The boundary conditions (14) become now after expansion

$$\bar{\phi}_0 - \bar{\psi}_0 + \lambda[(\bar{\dot{\phi}}_0 - \bar{\dot{\psi}}_0)\xi_1 + \bar{\phi}_1 - \bar{\psi}_1] + \ldots = 0,$$
$$\bar{\dot{\phi}}_0 - \bar{\dot{\psi}}_0 + \lambda[(\bar{\dot{\phi}}_0 - \bar{\dot{\psi}}_0)\xi_1 + \bar{\dot{\phi}}_1 - \bar{\dot{\psi}}_1] + \ldots = 0.$$

Since the coefficients of the different powers of λ must disappear, these conditions split up into an infinite series of pairs of conditions. The first pair is

$$(26) \quad . \quad . \quad . \qquad \bar{\phi}_0 - \bar{\psi}_0 = 0, \qquad \bar{\dot{\phi}}_0 - \bar{\dot{\psi}}_0 = 0.$$

The second pair may be written

$$(27) \quad . \quad . \quad . \quad \bar{\phi}_1 - \bar{\psi}_1 = 0, \qquad \dot{\phi}_1 - \dot{\bar{\psi}}_1 = 0.$$

For, since $\bar{\chi}_0 = 0$, it follows from the equations (21) and (26) that $\bar{\phi}_0 - \bar{\psi}_0 = 0$.

I pass on to the condition (15). It is valid for all values of λ. We have thus

$$(28) \quad . \quad . \quad . \quad . \quad \phi_0 = 1, \qquad \text{when } \xi = 0;$$

$$(29) \quad . \quad . \quad . \quad . \quad \phi_1 = 0, \qquad \text{when } \xi = 0,$$

and so on.

Finally from (16) and (17) for great values of ξ

$$(30) \quad . \quad . \quad . \quad \psi_0 = -\gamma_0 + \frac{\delta_0}{\xi} + \frac{1}{\xi^3} P_0\left(\frac{1}{\xi}\right),$$

$$(31) \quad . \quad . \quad \psi_1 = \tfrac{1}{6}\xi^2[1 - X_2(u)] - \gamma_1 + \frac{\delta_1}{\xi} + \frac{1}{\xi^3} P_1\left(\frac{1}{\xi}\right),$$

and so on.

$X_2(u) = \dfrac{3u^2 - 1}{2}$ is Legendre's polynomial of the second degree.

4. *Radiative Equilibrium without Rotation.* — It follows from spherical symmetry that the system (21) may be written

$$(32) \quad . \quad . \quad . \quad . \quad \ddot{\phi}_0 + \frac{2}{\xi}\dot{\phi}_0 + \chi_0 = 0,$$

$$(33) \quad . \quad . \quad . \quad . \quad \ddot{\psi}_0 + \frac{2}{\xi}\dot{\psi}_0 = 0.$$

χ_0 is given as a function of ϕ_0 by the equation (18).

Of the conditions (26), (28), and (30), the latter involves no restriction, for the general solution of (33) is of the form

$$(34) \quad . \quad . \quad . \quad . \quad \psi_0 = -\gamma_0 + \frac{\delta_0}{\xi},$$

where γ_0 and δ_0 are constants. Instead of the ineffective condition (30) now appears the condition

$$(35) \quad . \quad . \quad . \quad . \quad \dot{\phi}_0 = 0, \qquad \text{when } \xi = 0.$$

This is a result of the assumed spherical symmetry (when $\lambda = 0$), and of the fact that the derivatives of the first order of the internal potential are continuous within the body.

The two conditions (28) and (35) fully define the function ϕ_0. For a numerically given κ, ϕ_0 can always be calculated to any degree of accuracy.

When $\bar{\phi}_0$ and $\dot{\phi}_0$ have been calculated, the constants γ_0 and δ_0 in the general expression (34) for ψ_0 can be deduced from the two remaining

conditions (26). We find immediately from the second of these conditions

$$(36) \quad \delta_0 = -\phi_0' \cdot \xi^2,$$

and then from the first

$$(37) \quad \gamma_0 = -\phi_0 \xi.$$

The differential equation (32) may be written

$$(38) \quad \frac{d^2}{d\xi^2}(\xi\phi_0) = -\xi\chi_0.$$

It is of the type

$$(39) \quad \ddot{\eta} = f(\xi, \eta).$$

It is shown in the appendix how such an equation can be numerically integrated along the real axis of ξ, if the function η is known in the immediate neighbourhood of $\xi = 0$.

It is easy to expand ϕ_0 in a series of ascending powers of ξ^2. The coefficients of the series are certain functions of κ. But instead of κ we introduce the parameter

$$(40) \quad \alpha = b\rho_c,$$

where ρ_c is the density at the centre of gravity of the gaseous mass. Then, from (5), (9), and (10)

$$(41) \quad \kappa = \frac{\alpha(1 - \frac{3}{4}\alpha)^3}{(1-\alpha)^4}.$$

In the expansions

$$(42) \quad \chi_0 = a_0 + a_1\xi^2 + a_2\xi^4 + a_3\xi^6 + \ldots,$$

$$(43) \quad \phi_0 = 1 + b_1\xi^2 + b_2\xi^4 + b_3\xi^6 + \ldots$$

the coefficients a_ν and b_ν are functions of α.

By the introduction of (42) and (43) into (38) the relation

$$(44) \quad a_{\nu-1} = -2\nu(2\nu+1)b_\nu, \qquad \nu = 1, 2, 3, \ldots$$

is obtained. Finally, after these expressions for $a_{\nu-1}$ have been introduced in (42), the equation (18) gives successively expressions for the coefficients $b_1, b_2, \ldots$ The results are

$$(45) \quad \begin{cases} b_1 = -\frac{1}{6}\frac{(1-\alpha)^4}{(1-\frac{3}{4}\alpha)^3}, \\ b_2 = +\frac{1}{40}\frac{(1-\alpha)^9}{(1-\frac{3}{4}\alpha)^5}, \\ b_3 = -\frac{1}{5040}\frac{(1-\alpha)^{13}}{(1-\frac{3}{4}\alpha)^7}(19 - 54\alpha), \\ b_4 = +\frac{1}{1088640}\frac{(1-\alpha)^{17}}{(1-\frac{3}{4}\alpha)^9}(619 - 4836\alpha + 6912\alpha^2), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

For a perfect gas κ and α vanish ; then

$$(46) \quad . \qquad \phi_0 = 1 - \frac{1}{6}\xi^2 + \frac{1}{40}\xi^4 - \frac{19}{5040}\xi^6 + \frac{619}{1088640}\xi^8 - \ . \ . \ .$$

Numerical integration of (38) has been carried out for $\alpha = 0$. The numerical values obtained for ϕ_0 and $\chi_0 = \phi_0^3$ appear in columns 2 and 3 of the table on p. 701. By interpolation and by numerical derivation is obtained

$$(47) \quad . \quad . \quad . \quad \bar{\xi} = 6{\cdot}89691, \qquad \bar{\phi}_0 = -0{\cdot}042428.$$

Finally, in consequence of (36) and (37),

$$(48) \quad . \quad . \quad . \quad \delta_0 = +2{\cdot}0182, \qquad \gamma_0 = +0{\cdot}29262.$$

5. *Radiative Equilibrium of a Slightly Oblate Rotating Star.*—I pass on to study the differential equations (22). Using the formula (19) and the notation (20) these differential equations become

$$(49) \quad . \quad . \quad . \quad . \quad \nabla\phi_1 + \mathrm{E}(\xi, \alpha)\phi_1 = 1,$$

$$(50) \quad . \quad . \quad . \quad . \quad \nabla\psi_1 = 1.$$

The function $\mathrm{E}(\xi, \alpha)$ may be regarded as numerically known within the limits $0 < \xi < \bar{\xi}(\alpha)$. For small values of ξ, E may be expanded according to powers of ξ^2. The coefficients of the expansion are rational functions of α. We write

$$(51) \quad . \quad . \quad . \quad \mathrm{E} = e_0 + e_1\xi^2 + e_2\xi^4 + e_3\xi^6 + \ . \ . \ .$$

From the definition (20) and the formulæ (42), (44), and (45) we get, after some calculation, the following expressions for the first coefficients e_ν

$$(52) \quad . \quad \begin{cases} e_0 = +3\dfrac{(1-\alpha)^5}{(1-\frac{3}{4}\alpha)^2}, \\ e_1 = -\dfrac{1}{2}\dfrac{(1-\alpha)^9}{(1-\frac{3}{4}\alpha)^4}(2-9\alpha), \\ e_2 = +\dfrac{1}{120}\dfrac{(1-\alpha)^{13}}{(1-\frac{3}{4}\alpha)^6}(28-399\alpha+756\alpha^2), \\ e_3 = -\dfrac{1}{15120}\dfrac{(1-\alpha)^{17}}{(1-\frac{3}{4}\alpha)^8}(720-21921\alpha+109890\alpha^2-129114\alpha^3), \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases}$$

The required solution of the system (49), (50) must satisfy the four conditions (27), (29), and (31).

The functions ϕ_1 and ψ_1 may be expanded in series of spherical harmonics. It might be assumed quite generally that ϕ_1 and ψ_1 depend not only on the latitude variable u but also on the longitude v. In other words, it would not be necessary to impose the condition that the required solution is symmetrical about the axis of rotation. We should

then have to introduce all the fundamental spherical harmonics. But when this is done, it will be seen that the terms depending on the longitude v drop out. In order to shorten the writing they are left out from the beginning. The series then proceed according to Legendre's polynomials $X_n(u)$ and take the form

$$(53) \quad \begin{cases} \phi_1 = \phi_{1.0} + \phi_{1.1}X_1(u) + \phi_{1.2}X_2(u) + \ldots, \\ \psi_1 = \psi_{1.0} + \psi_{1.1}X_1(u) + \psi_{1.2}X_2(u) + \ldots, \end{cases}$$

where $\phi_{1.n}$ and $\psi_{1.n}$ depend on ξ and a, but not on u and v.

These series should now be introduced into the differential equations (49) and (50). According to the theory of spherical harmonics the formula

$$(54) \quad \nabla[F(\xi)Z_n(u, v)] = D_nF(\xi)Z_n(u, v)$$

holds good; Z_n is an arbitrary spherical harmonic of order n; for the sake of brevity the notation

$$(55) \quad D_nF = \dot{F} + \frac{2}{\xi}\dot{F} - \frac{n(n+1)}{\xi^2}F$$

is used. Taking (54) into consideration we find that the partial differential equations (49) and (50) resolve themselves into the following pairs of ordinary differential equations

$$(56) \quad D_0\phi_{1.0} + E \cdot \phi_{1.0} = 1, \qquad D_0\psi_{1.0} = 1;$$

$$(57) \quad D_n\phi_{1.n} + E \cdot \phi_{1.n} = 0, \qquad D_n\psi_{1.n} = 0, \qquad n = 1, 2, 3, \ldots$$

In consequence of (27), (29), and (31) the required solution must satisfy the following conditions

$$(58) \quad \bar{\phi}_{1.n} - \bar{\psi}_{1.n} = 0, \qquad \dot{\phi}_{1.n} - \dot{\bar{\psi}}_{1.n} = 0, \qquad n = 0, 1, 2, \ldots$$

$$(59) \quad \phi_{1.n} = 0, \qquad \text{when } \xi = 0, \qquad n = 0, 1, 2, \ldots$$

$$(60) \quad \psi_{1.0} = \tfrac{1}{6}\xi^2 - \gamma_1 + \frac{\delta_1}{\xi} + \frac{1}{\xi^3}P_{1.0}\left(\frac{1}{\xi}\right),$$

$$(61) \quad \psi_{1.2} = -\tfrac{1}{6}\xi^2 + \frac{1}{\xi^3}P_{1.2}\left(\frac{1}{\xi}\right),$$

$$(62) \quad \psi_{1.n} = \frac{1}{\xi^3}P_{1.n}\left(\frac{1}{\xi}\right), \qquad n = 1, 3, 4, \ldots$$

In the three expressions (60), (61), and (62), which hold good for large values of ξ, the term in $\frac{1}{\xi^2}$ is lacking.

The equations for $\psi_{1.0}, \psi_{1.1}, \ldots$ can be integrated generally. We have as general solution of (56_2)

$$(63) \quad \psi_{1.0} = \tfrac{1}{6}\xi^2 - \gamma_1 + \frac{\delta_1}{\xi},$$

where γ_1 and δ_1 are constants. The condition (60) involves no limitation. Further we have as general solution of (57_2)

$$(64) \quad \psi_{1.n} = -\gamma_{1.n}\xi^n + \frac{\delta_{1.n}}{\xi^{n+1}}, \qquad n = 1, 2, 3, \ldots,$$

where $\gamma_{1.n}$ and $\delta_{1.n}$ are constants. The condition (62) shows that $\gamma_{1.1} = \delta_{1.1} = 0$, so that

$$(65) \quad \psi_{1.1} \equiv 0.$$

Further from (61) and (64)

$$(66) \quad \psi_{1.2} = -\tfrac{1}{6}\xi^2 + \frac{\delta_{1.2}}{\xi^3}.$$

Finally from (62) and (64)

$$(67) \quad \psi_{1.n} = \frac{\delta_{1.n}}{\xi^{n+1}}, \qquad n = 3, 4, \ldots$$

I pass on to the linear differential equation (56_1). It has a perfectly definite particular solution of the form

$$(68) \quad \phi_{1.0} = \frac{\xi^2}{6}[1 + k_1\xi^2 + k_2\xi^4 + k_3\xi^6 + k_4\xi^8 + \ldots].$$

From (51) and (52) we get after calculation the following expressions for the coefficients

$$(69) \begin{cases} k_1 = -\dfrac{3}{20}\dfrac{(1-\alpha)^5}{(1-\frac{3}{4}\alpha)^2}, \qquad k_2 = +\dfrac{1}{840}\dfrac{(1-\alpha)^9}{(1-\frac{3}{4}\alpha)^4}(29 - 99\alpha), \\ k_3 = -\dfrac{1}{60480}\dfrac{(1-\alpha)^{13}}{(1-\frac{3}{4}\alpha)^6}(409 - 3870\alpha + 6156\alpha^2), \\ k_4 = +\dfrac{1}{1584000}\dfrac{(1-\alpha)^{17}}{(1-\frac{3}{4}\alpha)^8}(1979 - 36051\alpha + 144180\alpha^2 - 148608\alpha^3). \end{cases}$$

For a perfect gas ($\alpha = 0$) we have

$$(70) \quad \phi_{1.0} = \frac{\xi^2}{6}\left[1 - \frac{3}{20}\xi^2 + \frac{29}{840}\xi^4 - \frac{409}{60480}\xi^6 + \frac{1979}{1584000}\xi^8 - \ldots\right].$$

The general solution of (56_1) is obtained by adding to the particular solution (68) the general solution of the homogeneous equation. But this homogeneous equation has a particular solution which is $= 1$ when $\xi = 0$, and another particular solution which is infinite as ξ^{-1} when $\xi = 0$. Since $\phi_{1.0} = 0$ when $\xi = 0$ because of (59), it will be seen that (68) is the required solution.

I proceed to the homogeneous linear equation (57_1). It has a particular solution of the form

$$(71) \quad f_n = \xi^n[1 + L_{n.1}\xi^2 + L_{n.2}\xi^4 + L_{n.3}\xi^6 + \ldots]$$

and another particular solution of the form

$$\xi^{-n-1}[1 + q_1\xi^2 + q_2\xi^4 + \ldots].$$

The general solution is a linear combination of the two. But, since $\phi_{1.n}$ must be finite ($=0$) when $\xi=0$,

$$(72) \quad \phi_{1.n} = -\epsilon_{1.n} f_n, \qquad n = 1, 2, 3, \ldots$$

where $\epsilon_{1.n}$ is a constant.

The coefficients in the expansion (71) are certain rational functions of a. In this paper we have only use of the coefficients

$$(73) \quad \begin{cases} L_{2.1} = -\frac{3}{14}\frac{(1-a)^5}{(1-\frac{3}{4}a)^2}, \\ L_{2.2} = +\frac{1}{504}\frac{(1-a)^9}{(1-\frac{3}{4}a)^4}(23-72a), \\ L_{2.3} = -\frac{1}{55440}\frac{(1-a)^{13}}{(1-\frac{3}{4}a)^6}(491-4258a+6462a^2), \\ \ldots\ldots\ldots\ldots \end{cases}$$

For a perfect gas ($a=0$) we have

$$(74) \quad f_2 = \xi^2\left[1 - \frac{3}{14}\xi^2 + \frac{23}{504}\xi^4 - \frac{491}{55440}\xi^6 + \ldots\right].$$

In the expressions (63), (66), (67), and (72) appear the coefficients γ_1, δ_1, $\delta_{1.2}$, $\ldots$ $\delta_{1.n}$, $\ldots$; $\epsilon_{1.1}$, $\epsilon_{1.2}$, $\ldots$ $\epsilon_{1.n}$, $\ldots$ They will be determined by the conditions (58). For $n=0$ these conditions give

$$(75) \quad \delta_1 = \tfrac{1}{3}\bar{\xi}^3 - \bar{\xi}^2\bar{\phi}_{1.0},$$

$$(76) \quad \gamma_1 = \tfrac{1}{2}\bar{\xi}^2 - \bar{\xi}\bar{\phi}_{1.0} - \bar{\phi}_{1.0}.$$

The condition corresponding to $n=1$ gives $\epsilon_{1.1}=0$, *i.e.*

$$(77) \quad \phi_{1.1} \equiv 0.$$

For $n=2$ the conditions (58) are of the form

$$\epsilon_{1.2}\bar{f}_2 + \delta_{1.2}\bar{\xi}^{-3} = \tfrac{1}{6}\bar{\xi}^2,$$
$$\epsilon_{1.2}\bar{f}_2' - 3\delta_{1.2}\bar{\xi}^{-4} = \tfrac{1}{3}\bar{\xi}.$$

For $n>2$ they are

$$\epsilon_{1.n}\bar{f}_n + \delta_{1.n}\bar{\xi}^{-n-1} = 0,$$
$$\epsilon_{1.n}\bar{f}_n' - (n+1)\delta_{1.n}\bar{\xi}^{-n-2} = 0.$$

The determinants

$$(78) \quad \bar{\Delta}_n = \bar{\xi}\bar{f}_n' + (n+1)\bar{f}_n, \qquad n = 2, 3, \ldots$$

are not zero. The proof is omitted here. Thus

$$(79) \quad \epsilon_{1.2} = \tfrac{5}{6}\bar{\xi}^2 \frac{1}{\bar{\xi}\bar{f}_2' + 3\bar{f}_2},$$

$$(80) \quad \delta_{1.2} = \tfrac{1}{6}\bar{\xi}^5 \frac{\bar{\xi}\bar{f}_2' - 2\bar{f}_2}{\bar{\xi}\bar{f}_2' + 3\bar{f}_2},$$

$$(81) \quad \epsilon_{1.n} = \delta_{1.n} = 0, \qquad n = 3, 4, \ldots$$

It follows from (81) that $\phi_{1.n} = \psi_{1.n} = 0$ $(n = 3, 4, 5, \ldots)$.

Finally, neglecting λ^2, the scalar gravity-potential ϕ is given by the formula

$$(82) \quad \phi = \phi_0(\xi) + \lambda\left\{\phi_{1.0}(\xi) - \tfrac{5}{6}\frac{\bar{\xi}^2}{\bar{\xi}\bar{f}_2 + 3\bar{f}_2} f_2(\xi) X_2(u)\right\} + \ldots$$

The differential equations (56_1) and (57_1) (for $n = 2$) may be written

$$(83) \quad \frac{d^2}{d\bar{\xi}^2}(\xi\phi_{1.0}) = -\mathrm{E}\cdot\xi\phi_{1.0} + \xi;$$

$$(84) \quad \frac{d^2}{d\bar{\xi}^2}(\xi f_2) = \left(\frac{6}{\bar{\xi}^2} - \mathrm{E}\right)\xi f_2 .$$

They are of the type (39). Thus the functions $\phi_{1.0}$ and f_2, given for small values of ξ by the expansions (68) and (71) (for $n = 2$), can be calculated for real values, or ξ by the use of the method given in the appendix (for every stated value of α).

The numerical integrations of (83) and (84) have been carried out for $\alpha = 0$ (perfectly gaseous stars). The numerical values of $\phi_{1.0}$ and f_2 are tabulated in columns 4 and 5 of the table * on p. 701. Using this table and the formulæ (75), (76), (79), and (80) the values

$$(85) \quad \begin{cases} \bar{\phi}_{1.0} = +\ 5{\cdot}83800, & \bar{\phi}_{1.0} = +\ 2{\cdot}03911 \\ \bar{f}_2 = +11{\cdot}2780, & \bar{f}_2 = +\ 3{\cdot}0409 \\ \delta_1 = +12{\cdot}361, & \gamma_1 = +\ 3{\cdot}8822 \\ \epsilon_{1.2} = +\ 0{\cdot}72326, & \delta_{1.2} = -75{\cdot}122 \end{cases}$$

have been obtained (for $\alpha = 0$).

6. *Mass, Mean Density, and External Shape.*—I return to the formula (82). The equation of the boundary $\phi = 0$, after solution gives ξ as a function of u at the surface of the star. Thus at the surface, since $\bar{\phi}_0 = 0$,

$$(86) \quad \frac{\xi - \bar{\xi}}{\bar{\xi}} = \frac{\lambda}{-\bar{\xi}\bar{\phi}_0}\left\{\bar{\phi}_{1.0} - \frac{5}{6}\frac{\bar{\xi}^2}{\bar{\zeta}_2 + 3} X_2(u)\right\} + \ldots$$

with the abbreviating notation

$$(87) \quad \bar{\zeta}_2 = \bar{\xi}\frac{\bar{f}_2}{\bar{f}_2}.$$

The volume N of the star is obtained from the formula

$$(88) \quad \begin{cases} \mathrm{N} = \dfrac{4\pi}{3} l^3 \bar{\xi}^3 + l^3 \displaystyle\int_0^{2\pi}\int_{-1}^{+1} (\xi - \bar{\xi})\bar{\xi}^2 du dv \\ \quad = \dfrac{4\pi}{3} l^3 \bar{\xi}^3 \left\{1 + \lambda \dfrac{3\bar{\phi}_{1.0}}{-\bar{\xi}\bar{\phi}_0} + \ldots\right\}. \end{cases}$$

* The last two columns in the table refer to a following paper treating the equilibrium of a double star system.

I proceed to introduce the mass M of the star as one of its fundamental parameters. From the expression (16), compared with the well-known expansion $V = \frac{GM}{r} + \ldots$ of the external attraction potential, we get the relation

$$(89) \quad h = \frac{M}{\delta};$$

(11) then gives

$$(90) \quad l^3 = \frac{b}{4\pi\kappa}\frac{M}{\delta}.$$

The quantity δ is a function of κ and λ. It is known to us only for small values of λ by the series (17_5).

After introducing the expressions (36) and (75) for δ_0 and δ_1 we get from (90)

$$(91) \quad M = -\frac{4\pi\kappa}{b} l^3 \cdot \bar{\phi}_0 \bar{\xi}^2 \cdot \left\{1 + \lambda \frac{\frac{1}{3}\bar{\xi}^2 - \bar{\xi}\bar{\phi}_{1\cdot 0}}{-\bar{\xi}\bar{\phi}_0} + \ldots\right\}.$$

Using (88) and (91) we find for the mean density ρ_m the expression

$$(92) \quad \rho_m = \frac{M}{N} = -\frac{3\bar{\phi}_0}{\bar{\xi}}\frac{\kappa}{b}\left\{1 - \frac{\lambda}{-\bar{\xi}\bar{\phi}_0}(\bar{\xi}\bar{\phi}_{1\cdot 0} + 3\bar{\phi}_{1\cdot 0} - \tfrac{1}{3}\bar{\xi}^2) + \ldots\right\}.$$

If only the principal term in this formula is retained we get, from the definition (12), the relation

$$(93) \quad \frac{\omega^2}{2\pi G\rho_m} = \frac{\bar{\xi}}{-3\bar{\phi}_0}\lambda + \ldots$$

Through the equations (92) and (93) the parameters κ and λ can be expressed as functions of ρ_m and ω.

I return to the formula (86) valid at the surface of the star. At the equator $u = 0$, $X_2(u) = -\frac{1}{2}$; at the poles, on the other hand, $u = 1$, $X_2(u) = +1$. Thus we get for the oblateness o of the boundary surface the general expression

$$(94) \quad o = \frac{5}{4}\frac{\bar{\xi}}{-\bar{\phi}_0}\frac{\lambda}{\bar{\zeta}_2 + 3} + \ldots = \frac{15}{4}\frac{1}{\bar{\zeta}_2 + 3}\frac{\omega^2}{2\pi G\rho_m} + \ldots$$

For perfectly gaseous stars $\frac{\kappa}{b} = \rho_c$. After introducing the numerical values of (47) and (85), the relations (93) and (92) and the expression (94) are simplified to

$$(95) \quad \lambda = 0{\cdot}0184550\frac{\omega^2}{2\pi G\rho_m} + \ldots,$$

$$(96) \quad \rho_c : \rho_m = 54{\cdot}1857\left\{1 + 0{\cdot}99153 \cdot \frac{\omega^2}{2\pi G\rho_m} + \ldots\right\},$$

$$(97) \quad o = 0{\cdot}77166 \cdot \frac{\omega^2}{2\pi G\rho_m} + \ldots$$

7. *Some Relations between the Parameters.*—The original parameters of the problem were β, $\tilde{\Phi}$, and ω. They can be expressed as functions of the fundamental parameters M, ρ_m, and ω.

If in the formula (8) we replace h and l by their expressions according to (89) and (90), we get as a generalised form of Eddington's quartic equation

$$(98) \quad . \quad . \quad . \quad . \quad \frac{1-\beta}{\beta^4} = \frac{\pi}{48}\, a \left(\frac{m}{R}\right)^4 G^3 \frac{M^2}{\delta^2}.$$

Using the development (17_5) for δ and the expressions (36) and (75) for δ_0 and δ_1 we find

$$(99) \quad . \quad \frac{1-\beta}{\beta^4} = \frac{\pi}{48}\, a \left(\frac{m}{R}\right)^4 \frac{G^3 M^2}{\bar{\xi}^4 \bar{\phi}_0^{\,2}} \left\{ 1 - 2\lambda \frac{\frac{1}{3}\bar{\xi} - \bar{\phi}_{1.0}}{-\bar{\phi}_0} + \ldots \right\},$$

and with regard to (93)

$$(99') \quad . \quad \frac{1-\beta}{\beta^4} = \frac{\pi}{48}\, a \left(\frac{m}{R}\right)^4 \frac{G^3 M^2}{\bar{\xi}^4 \bar{\phi}_0^{\,2}} \left\{ 1 - \frac{\omega^2}{2\pi G \rho_m} \left(2 - 6 \frac{\bar{\phi}_{1.0}}{\bar{\xi}} \right) + \ldots \right\}.$$

Numerical application gives for perfectly gaseous stars the equation

$$(100) \quad \frac{1-\beta}{\beta^4} = 0{\cdot}002668 m^4 \left(\frac{M}{M_\odot}\right)^2 \left\{ 1 - 0{\cdot}22607 \frac{\omega^2}{2\pi G \rho_m} + \ldots \right\}.$$

[In the calculation the following values were employed

$$(101) \quad . \quad . \quad . \quad . \quad \begin{cases} a = 7{\cdot}060 \,.\, 10^{-15} \\ R = 8{\cdot}2962 \,.\, 10^{7} \\ G = 6{\cdot}6576 \,.\, 10^{-8} \\ M_\odot = 1{\cdot}94302 \,.\, 10^{33} \end{cases}$$

together with the values (47) and (85)]. For slightly oblate perfectly gaseous stars β increases with ω, when M and ρ_m are constant.

It remains to consider the values $\tilde{\Phi}$ and Φ_c of the gravity-potential Φ at the boundary and at the centre of gravity. Using (4_1), (5), (89), and (90) we obtain

$$\Phi_c - \tilde{\Phi} = G\frac{h}{l} = \frac{GM}{l}\,\frac{1}{\delta} = G\left(\frac{4\pi\kappa}{b}\right)^{1/3}\left(\frac{M}{\delta}\right)^{2/3}.$$

The expression (16), compared with (4_2), then gives the value of the gravity-potential at the surface as

$$\tilde{\Phi} = G\frac{h}{l}\cdot\gamma = \frac{GM}{l}\,\frac{\gamma}{\delta} = G\left(\frac{4\pi\kappa}{b}\right)^{1/3}\left(\frac{M}{\delta}\right)^{2/3}\cdot\gamma.$$

Thus Φ_c and $\tilde{\Phi}$ are expressed as functions of M, κ, and λ. Evidently they can also be expressed by M, ρ_m, and ω.

8. *The Variation of Gravity on the Surface of the Star.*—For gravity, if λ^2 is neglected, the expression

$$\frac{d\Phi}{dn_i} = -\frac{d\Phi}{dr} + \ldots = -\frac{Gh}{l^2}\tilde{\phi}$$

$$= \frac{Gh}{l^2}\{-\phi_0 - \lambda\phi_1 - \ldots\}$$

holds good. At the boundary

$$-\phi_0 = -\mathring{\phi}_0 + \bar{\phi}_0\lambda\frac{\bar{\phi}_1}{\mathring{\phi}_0} + \ldots = -\mathring{\phi}_0 - \lambda\frac{2\bar{\phi}_1}{\bar{\xi}} + \ldots$$

Thus gravity at the surface is

$$\frac{d\tilde{\Phi}}{dn_i} = -\mathring{\phi}_0\frac{Gh}{l^2}\left\{1 + \frac{\lambda}{\bar{\xi}\mathring{\phi}_0}(\bar{\xi}\mathring{\phi}_1 + 2\bar{\phi}_1) + \ldots\right\}.$$

Using (88) and (91) and writing in the latter h instead of $4\pi\kappa b^{-1}l^3$ we get

$$\frac{M}{N^{2/3}} = M^{1/3}\rho_m^{2/3} = -\left(\frac{3}{4\pi}\right)^{2/3}\cdot\mathring{\phi}_0\cdot\frac{h}{l^2}\left\{1 + \frac{\lambda}{-\bar{\xi}\mathring{\phi}_0}(\tfrac{1}{5}\bar{\xi}^2 - \bar{\xi}\mathring{\phi}_{1\cdot0} - 2\bar{\phi}_{1\cdot0}) + \ldots\right\}.$$

The first formula is divided by the latter. Instead of λ we introduce its expression according to (93). Further, we introduce

$$\bar{\xi}(\mathring{\phi}_1 - \mathring{\phi}_{1\cdot0}) + 2(\bar{\phi}_1 - \bar{\phi}_{1\cdot0}) = (\bar{\xi}\mathring{\phi}_{1\cdot2} + 2\bar{\phi}_{1\cdot2})X_2(u)$$

$$= -\epsilon_{1\cdot2}(\bar{\xi}\mathring{f}_2 + 2\bar{f}_2)X_2(u) = -\tfrac{5}{6}\bar{\xi}^2\frac{\bar{\zeta}_2 + 2}{\bar{\zeta}_2 + 3}X_2(u).$$

Thus we get finally

$$(102)\quad \frac{d\tilde{\Phi}}{dn_i} = \left(\frac{4\pi}{3}\right)^{2/3}GM^{1/3}\rho_m^{2/3}\left\{1 - \frac{\omega^2}{2\pi G\rho_m}\left[1 - \frac{5}{2}\frac{\bar{\zeta}_2 + 2}{\bar{\zeta}_2 + 3}X_2(u)\right] + \ldots\right\}$$

For a perfectly gaseous star we have to introduce here the numerical value

$$(103)\quad \ldots\quad \frac{5}{2}\frac{\bar{\zeta}_2 + 2}{\bar{\zeta}_2 + 3} = 1{\cdot}9856.$$

9. *Surface Brightness and Effective Temperature.*—The quantity of energy which radiates per second from the unit area of the star's surface is called its surface brightness. It is denoted by $\tilde{F}$. We have shown [I, (36)] that

$$(104)\quad \ldots\quad \tilde{F} = \frac{c}{C}\frac{d\tilde{\Phi}}{dn_i}.$$

$c = 3\cdot10^{10}$ is the velocity of light. The quantity C, which is constant within the star, appeared at first in the formula [I. (33)]. Its expression

depends on the accepted law of absorption. I will suppose in accordance with Eddington that the absorption coefficient k is given by the formula

$$(105) \quad . \quad . \quad . \quad . \quad . \quad k = \mathrm{K}p\mathrm{T}^{-4},$$

where K is a constant. Then the expression of C is [I. (44)]

$$(106) \quad . \quad . \quad . \quad . \quad \mathrm{C} = \frac{a}{3}\mathrm{K}\frac{\beta}{(1-\beta)^2}.$$

Thus in case of the assumption (105) the surface brightness is given by

$$(107) \quad . \quad . \quad . \quad . \quad \tilde{\mathrm{F}} = \frac{3c}{a\mathrm{K}}\frac{(1-\beta)^2}{\beta}\frac{d\tilde{\Phi}}{dn_i}.$$

The effective temperature T_e of the star is the temperature of the black body having the same surface brightness. In accordance with Stefan's law

$$\mathrm{T}_e{}^4 = \frac{1}{\pi\mu}\tilde{\mathrm{F}} = \frac{4}{ac}\tilde{\mathrm{F}}$$

we have

$$\mathrm{T}_e{}^4 = \frac{12}{a^2\mathrm{K}}\frac{(1-\beta)^2}{\beta}\frac{d\tilde{\Phi}}{dn_i}.$$

Introducing the expression (102) we obtain T_e as function of u and the fundamental parameters M, ρ_m, and ω.

For a perfectly gaseous star with slight oblateness the effective temperature is given by the formula

$$(108) \quad . \quad \mathrm{T}_e = \mathrm{K}_1(1-\beta)^{1/2}\beta^{-1/4}\left(\frac{\mathrm{M}}{\mathrm{M}_\odot}\right)^{1/12}\rho_m{}^{1/6}$$

$$\left\{1 - \frac{\omega^2}{2\pi\mathrm{G}\rho_m}[0{\cdot}25 - 0{\cdot}49639\mathrm{X}_2(u)] + \ldots\right\},$$

where

$$\mathrm{K}_1 = \left(\frac{4\pi}{3}\right)^{1/6} . \mathrm{G}^{1/4} . 12^{1/4} . a^{-1/2} . \mathrm{K}^{-1/4} . \mathrm{M}_\odot{}^{1/12}.$$

The value of K can be deduced by the well-determined total luminosity L of Capella using the formula [I, (52)].

The effective temperature has its maximum value at the poles and its minimum value at the equator.

Appendix, Numerical Integration.—In the foregoing pages three differential equations (38), (83), and (84) have appeared, all of the type (39). For their numerical integration the following general method may be used.

The required solution of (39) may have the initial values $\eta = b$, $\dot{\eta} = \dot{b}$,

when $\xi = 0$. If we suppose this solution introduced into $f(\xi, \eta)$ we get a function of ξ which is denoted by $f(\xi)$. Clearly

$$\eta(\xi) = b + \dot{b}\xi + \int_0^{\xi}\int_0^{v} f(u)dudv.$$

We consider equidistant values of ξ: $(0, \omega, 2\omega, \ldots)$ and imagine the corresponding values $\eta(i\omega)$ and their differences of the first and second order $\eta^1(\overline{i+\tfrac{1}{2}}\omega)$ and $\eta^2(i\omega)$. We have

$$\eta^1(\overline{i-\tfrac{1}{2}}\omega) = \eta(i\omega) - \eta(\overline{i-1}\omega)$$

$$= \dot{b}\omega + \int_{\overline{i-1}\omega}^{i\omega}\int_0^{v} f(u)dudv = \dot{b}\omega + \int_{\overline{i-\frac{1}{2}}\omega}^{\overline{i+\frac{1}{2}}\omega}\int_0^{v'-\frac{1}{2}\omega} f(u)dudv',$$

$$\eta^1(\overline{i+\tfrac{1}{2}}\omega) = \eta(\overline{i+1}\omega) - \eta(i\omega)$$

$$= \dot{b}\omega + \int_{i\omega}^{\overline{i+1}\omega}\int_0^{v} f(u)dudv = \dot{b}\omega + \int_{\overline{i-\frac{1}{2}}\omega}^{\overline{i+\frac{1}{2}}\omega}\int_0^{v'+\frac{1}{2}\omega} f(u)dudv'.$$

Thus we get for the difference of second order the expression

$$\eta^2(i\omega) = \eta^1(\overline{i+\tfrac{1}{2}}\omega) - \eta^1(\overline{i-\tfrac{1}{2}}\omega) = \int_{\overline{i-\frac{1}{2}}\omega}^{\overline{i+\frac{1}{2}}\omega}\int_{v'-\frac{1}{2}\omega}^{v'+\frac{1}{2}\omega} f(u)dudv'.$$

Into this we now introduce new integration variables, t and t', by putting $v' = (i+t')\omega$, $u = v' + t\omega = (i+t+t')\omega$. We then get

$$(109) \quad . \qquad \eta^2(i\omega) = \omega^2\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} f(\overline{i+t+t'}\omega)dtdt'.$$

We now consider the unknown values $f(i\omega) = f[i\omega, \eta(i\omega)]$ and their differences of the first, second, etc. order. As usual we use the notation

$$f^1(i\omega) = \tfrac{1}{2}[f^1(\overline{i-\tfrac{1}{2}}\omega) + f^1(\overline{i+\tfrac{1}{2}}\omega)],$$

$$f^3(i\omega) = \tfrac{1}{2}[f^3(\overline{i-\tfrac{1}{2}}\omega) + f^3(\overline{i+\tfrac{1}{2}}\omega)].$$

According to Stirling's formula of interpolation we have

$$f(\overline{i+t+t'}\omega) = f(i\omega) + (t+t')f^1(i\omega) + \frac{(t+t')^2}{2!}f^2(i\omega)$$

$$+ \frac{(t+t')[(t+t')^2-1]}{3!}f^3(i\omega) + \frac{(t+t')^2[(t+t')^2-1]}{4!}f^4(i\omega) + \ldots$$

This development is introduced into (109). Integration is performed term by term. For it we use the formula

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}}(t+t')^p dtdt' = \frac{1+(-1)p}{(p+1)(p+2)}.$$

Thus, finally, (109) leads to the difference equation

$$(110)\ .\qquad \eta^2(i\omega)=\omega^2\{f(i\omega)+\tfrac{1}{12}f^2(i\omega)-\tfrac{1}{240}f^4(i\omega)+\ .\ .\ .\}.$$

By means of this difference formula the equivalent differential equation (39) can be numerically integrated. Let us suppose that the values of η, and so also of f, corresponding to $\xi=0$ and $\xi=\omega$ are known, *e.g.* by a series of powers in the neighbourhood of $\xi=0$. I will show that the formula (110) leads successively to knowledge of the values of the unknown quantities for $\xi=2\omega,\ 3\omega,\ .\ .\ .\ .$ For simplicity's sake I will assume that ω is so small that the fourth difference $f^4(i\omega)$ in (110) can be neglected. Let us assume generally that $\eta(0),\ \eta(\omega),\ .\ .\ .\ \eta(i\omega)$ have already been calculated and that it is desired to find $\eta(\overline{i+1}\omega)$. We may construct the following table

$\eta(0)$	$\eta^1(\frac{1}{2}\omega)$		$f(0)$	$f^1(\frac{1}{2}\omega)$		$\omega^2 f(0)$	
$\eta(\omega)$		$\eta^2(\omega)$	$f(\omega)$		$f^2(\omega)$	$\omega^2 f(\omega)$	$\frac{\omega^2}{12}f^2(\omega)$
	$\eta^1(\frac{3}{2}\omega)$			$f^1(\frac{3}{2}\omega)$			
$\eta(2\omega)$		$\eta^2(2\omega)$	$f(2\omega)$		$f^2(2\omega)$	$\omega^2 f(2\omega)$	$\frac{\omega^2}{12}f^2(2\omega)$
.	.		.	.	.	.	
.	.		.	.	.	.	
.	.		.	.	.	.	
	$\eta^1(\overline{i-\frac{3}{2}}\omega)$			$f^1(\overline{i-\frac{3}{2}}\omega)$			
$\eta(\overline{i-1}\omega)$		$\eta^2(\overline{i-1}\omega)$	$f(\overline{i-1}\omega)$		$f^2(\overline{i-1}\omega)$	$\omega^2 f(\overline{i-1}\omega)$	$\frac{\omega^2}{12}f^2(\overline{i-1}\omega)$
$\eta(i\omega)$	$\eta^1(\overline{i-\frac{1}{2}}\omega)$		$f(i\omega)$	$f^1(\overline{i-\frac{1}{2}}\omega)$		$\omega^2 f(i\omega)$	

By extrapolation in the last column we get an approximate value for $\frac{\omega^2}{12}f^2(i\omega)$. From it we calculate, with the help of (110), an approximation for $\eta^2(i\omega)$, which is introduced into the third column. Then we can provisionally add one line to the second and first columns also. From the approximate value of $\eta(\overline{i+1}\omega)$ we calculate $f(\overline{i+1}\omega)=f[\overline{i+1}\omega,\ \eta(\overline{i+1}\omega)]$ and introduce it provisionally into the fourth column. Then the fifth, sixth, and seventh columns can be provisionally continued one line further. Finally, we get a definitive value for $\frac{\omega^2}{12}f^2(i\omega)$ which only deviates slightly from the extrapolated value. With this definitive value the same operation is repeated. Thus the definitive values of $\eta^2(i\omega)$, $\eta^1(\overline{i+\frac{1}{2}}\omega)$, $\eta(\overline{i+1}\omega)$, $f(\overline{i+1}\omega)$, $f^1(\overline{i+\frac{1}{2}}\omega)$, $f^2(i\omega)$, and $\omega^2 f(\overline{i+1}\omega)$ are obtained. They differ only by a few units in the last decimal place from the provisional values. By this process a line is added to the table. The double calculation provides a valuable means of control.

In applying this method to the differential equations (38), (83), and (84) with $\kappa=0$, I calculated with 5 or 6 decimal places. From $\xi=0$ to $\xi=2{\cdot}5$ I chose $\omega=\frac{1}{8}$. Then from $\xi=2{\cdot}0$ to $\xi=7{\cdot}5$ I put $\omega=\frac{1}{4}$. The results are tabulated on p. 701 in the first five columns. The last two columns refer to a paper which follows treating the equilibrium figures of double stars.

ξ.	ϕ_0.	ϕ_0^2.	$\phi_{1\cdot0}$.	f_2.	f_3.	f_4.
0·00	+1·00000	+1·00000	0·000000	0·000000	0·000000	0·000000
0·25	0·98968	0·96936	10320	61674	15464	3873
0·50	95984	88429	40190	23729	12005	60470
0·75	91354	76240	86762	50202	38644	16570
1·00	85505	62514	14647	82389	86199	88643
1·25	78897	49112	21613	1·1725	1·5697	2·0462
1·50	71949	37246	29364	1·5251	2·5165	3·9977
1·75	64998	27460	37823	1·8681	3·7024	6·9753
2·00	58284	19800	47023	2·1964	5·1293	11·229
2·25	51961	14030	57081	2·5108	6·8045	17·034
2·50	46112	98048	68159	2·8155	8·7432	24·702
2·75	40767	67750	80440	3·1167	10·967	34·587
3·00	35922	46354	94110	3·4204	13·506	47·093
3·25	31554	31416	1·0935	3·7326	16·393	62·677
3·50	27626	21084	1·2632	4·0586	19·665	81·854
3·75	24098	13993	1·4518	4·4030	23·363	105·19
4·00	20928	91660	1·6606	4·7695	27·529	133·33
4·25	18077	59067	1·8908	5·1612	32·208	166·94
4·50	15507	37287	2·1435	5·5807	37·444	206·79
4·75	13185	22922	2·4197	6·0301	43·284	253·67
5·00	11082	13610	2·7202	6·5112	49·774	308·44
5·25	91708	7713	3·0456	7·0251	56·959	372·03
5·50	74287	4100	3·3965	7·5730	64·886	445·40
5·75	58355	1987	3·7733	8·1555	73·598	529·58
6·00	43740	837	4·1762	8·7731	83·140	625·62
6·25	30287	278	4·6055	9·4260	93·552	734·64
6·50	17868	57	5·0611	10·1141	104·876	857·77
6·75	+ 6369	+ 3	5·5428	10·8371	117·148	996·18
7·00	− 4309	− 1	6·0503	11·5943	130·402	1151·07
7·25	− 14251	− 29	6·5832	12·3850	144·669	1323·64
7·50	− 23530	− 130	7·1406	13·2078	159·974	1515·09

Radiative Equilibrium of a Double-Star System with nearly Spherical Components. By H. v. Zeipel, Assoc.R.A.S.

1. *Introduction and Summary.*—The general theory * for the radiative equilibrium of gaseous masses can be carried out completely with the help of the spherical harmonics when the system consists of one or more bodies of almost spherical form. In the previous paper † the theory of a slightly oblate rotating star was given. In this paper I will study radiative equilibrium of a double-star system with circular orbit and constrained rotation, on the assumption that the mean radii of the stars are small fractions of the distance between them. The problem can, in fact, be fully solved by developments according to powers of these two small fractions. But it is considerably simplified if terms of the sixth order are neglected. Then the equilibrium of each star can be studied separately, as if the other component were reduced to a point.

We use this approximation to study analytically the form of each component, and also the distribution of brightness and effective temperature on its surface.

The results arrived at may be used as the starting-point of a theoretical deduction of the variability of certain double stars.

2. *The Conditions of Equilibrium.*—A double-star system, whose components have masses M and M′ (expressed in grammes), is assumed to rotate as a rigid body with angular velocity $\omega\left(\frac{\text{radians}}{\text{seconds}}\right)$ about an axis through the common centre of gravity; the axis of rotation is at right angles to the line connecting the centres of gravity of the two components; the distance R (in cms.) between the two centres of gravity is constant.

The system is referred to axis of co-ordinates taking part in the rotation. Hence arises a centrifugal potential $\Omega=\frac{\omega^2}{2}s^2$, where s is the distance from the axis of rotation.

U and V denote the inner and the outer attraction potentials of the first component; U′ and V′ the corresponding potentials of the second.

In the interior of the body M the gravity potential is

$$(1) \quad . \quad . \quad . \quad . \qquad \Phi = U + V' + \Omega.$$

Between the density ρ and the gravity-potential Φ in the first component there is, according to the general theory of radiative equilibrium, a relation of the form

$$(2) \quad . \quad . \qquad \frac{b\rho(1-\frac{3}{4}b\rho)^3}{(1-b\rho)^4} = \frac{ab}{3.4^3}\left(\frac{m}{R}\right)^4\frac{\beta^4}{1-\beta}(\Phi-\Phi_0)^3.$$

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* *Monthly Notices of R.A.S.*, **84**, 665 (here denoted by I).
† *Ibid.*, 684 (here denoted by II).

Here $1/b$ means the density of the gas at greatest possible compression, according to v. d. Waals' formula. The quantity a is a constant

$$(3) \quad . \quad . \quad . \quad . \quad a = 7{\cdot}060 \,.\, 10^{-15}.$$

The gas constant R has the value

$$(4) \quad . \quad . \quad . \quad . \quad \mathsf{R} = 8{\cdot}2962 \,.\, 10^{7}.$$

The mean molecular weight m of the gas has a value nearly $= 2$. $1 - \beta$ is the constant ratio between radiation force and gravity within the first component. Finally, $\tilde{\Phi}$ is the constant value of the gravity potential at the surface of this component.

Formulæ analogous to (1) and (2) apply also to the second component. But there the parameters corresponding to β and $\tilde{\Phi}$ have other values.

The potentials satisfy the equations of Poisson and Laplace

$$(5) \quad . \quad . \quad . \quad \Delta U + 4\pi G\rho = 0, \qquad \Delta V = 0,$$

where the attraction constant G has the value

$$(6) \quad . \quad . \quad . \quad . \quad G = 6{\cdot}6576 \,.\, 10^{-8}.$$

The potentials U′ and V′ of the second component satisfy similar equations.

The conditions attached to the solution required express the characteristic properties of the attraction potentials: the inner potential is finite within the body; at infinite distance the outer potential is reduced to M/r; at the surface of each component the values of its inner and outer potentials coincide; and this is true also of their derivatives of the first order.

3. *The Reduction of the Second Component to an Attracting Point.*— In what follows it is assumed that the distance R between the centres of gravity of the two stars is great in proportion to the mean radii of the stars.

The potential of a body can at a great distance be expanded according to negative powers of the radius vector. Thus

$$(7) \quad . \quad . \quad V' = G\left\{\frac{M'}{r'} + \frac{A' + B' + C' - 3I'}{2r'^3} + \ldots\right\}.$$

The term in r'^{-2} is lacking if the radius vector r' is reckoned from the centre of gravity of the body M.′ A′, B′, C′ denote the principal moments of inertia of the body M′ at its centre of gravity; I′ denotes the moment of inertia about the radius vector. These moments of inertia are of the order of magnitude $M'a'^2$, where a' is the mean radius of the body M′. From what follows it will be seen that the differences $A' - I'$, $B' - I'$, $C' - I'$ are of the same order of magnitude as $M'a'^2\left(\frac{a'}{R}\right)^3$.

We need only know the outer potential V′ inside or near the surface

of the first component, where $r' \propto R$. In that case the second term written in (7) is of the order of magnitude

$$(8) \quad . \quad . \quad . \quad . \quad . \quad G\frac{M'}{a'}\left(\frac{a'}{R}\right)^6.$$

Terms of this order of magnitude will be neglected. With this approximation the potential of M′ within and near the body M is expressed by

$$(9) \quad . \quad . \quad . \quad . \quad V' = G\frac{M'}{r'} + \ldots .$$

This leads to a formal simplification of the problem. The effect of the second component becomes the same as if its mass were concentrated in one point. Our task is reduced to studying the equilibrium of a star M describing, with constrained rotation, a circular orbit under the influence of the attraction from a point of mass M′ at a distance R.

There is a relation between the angular velocity ω and the distance R. When R is great in proportion to the mean radii of the stars, Kepler's third law applies. We may thus put

$$(10) \quad . \quad . \quad . \quad . \quad \omega^2 = \frac{G(M+M')}{R^3}(1+\sigma),$$

where σ is a small quantity which vanishes with the ratios between the mean radii of the stars and the distance R.

4. *The Centrifugal Potential.*—The origin of the moving axis of co-ordinates is chosen at the centre of gravity of M; the axis of z is parallel to the axis of rotation; the axis of x is directed towards the centre of gravity of M′.

We put

$$(11) \quad . \quad . \quad . \quad . \quad . \quad q = \frac{M'}{M+M'}.$$

The centrifugal potential is

$$\Omega = \frac{\omega^2}{2}\{(x-qR)^2+y^2\} = \frac{\omega^2 R^2}{2}\left(q^2 - 2q\frac{x}{r}\frac{r}{R}\right) + \frac{\omega^2}{2}(x^2+y^2).$$

Introducing polar co-ordinates and spherical harmonics, we have

$$(12) \quad . \quad . \quad . \quad \begin{cases} \dfrac{x}{r} = \sqrt{1-u^2}\cos v = X_1^{(1)}(u)\cos v, \\ \dfrac{y}{r} = \sqrt{1-u^2}\sin v = X_1^{(1)}(u)\sin v, \\ \dfrac{z}{r} = u = X_1(u); \\ \dfrac{x^2+y^2}{r^2} = 1-u^2 = \dfrac{2}{3}\left(1 - \dfrac{3u^2-1}{2}\right) = \dfrac{2}{3}[1 - X_2(u)]. \end{cases}$$

u denotes the cosinus of the angle between the radius vector and the axis of rotation, v the longitude reckoned from the direction towards M′.

Using the expression (10) for ω^2, the centrifugal potential as function of the polar co-ordinates is of the form

$$(13) \quad . \quad \Omega = \frac{GM'}{2R}(1+\sigma)\left\{q - 2\frac{r}{R}X_1^{(1)}(u)\cos v\right\} + \frac{\omega^2}{3}r^2[1 - X_2(u)].$$

5. *The Potential of the Disturbing Mass Point.*—The distance of the point (x, y, z) from the disturbing point (R, 0, 0) is

$$r' = \sqrt{(x-R)^2 + y^2 + z^2} = R\left(1 - 2\frac{x}{r}\frac{r}{R} + \frac{r^2}{R^2}\right)^{1/2}.$$

The potential of the point M′ at the point (x, y, z) in the neighbourhood of or within M can thus be developed in the form

$$(14) \quad . \quad . \quad . \quad V' = G\frac{M'}{r'} = G\frac{M'}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n X_n\left(\frac{x}{r}\right),$$

where X_n are Legendre's polynomials.

Introducing

$$\frac{x}{r} = \sqrt{1-u^2}\cos v$$

for α in the well-known expressions

$$X_1(\alpha) = \alpha, \quad X_2(\alpha) = \frac{3\alpha^2 - 1}{2}, \quad X_3(\alpha) = \frac{5\alpha^3 - 3\alpha}{2}, \quad X_4(\alpha) = \frac{35\alpha^4 - 30\alpha^2 + 3}{8},$$

and expanding according to fundamental spherical harmonics, we find

$$(15) \quad \begin{cases} X_1\left(\frac{x}{r}\right) \equiv Y_1(u, v) = X_1^{(1)}(u)\cos v, \\ X_2\left(\frac{x}{r}\right) \equiv Y_2(u, v) = -\frac{1}{2}X_2(u) + \frac{1}{4}X_2^{(2)}(u)\cos 2v, \\ X_3\left(\frac{x}{r}\right) \equiv Y_3(u, v) = -\frac{1}{4}X_3^{(1)}(u)\cos v + \frac{1}{2\,4}X_3^{(3)}(u)\cos 3v, \\ X_4\left(\frac{x}{r}\right) \equiv Y_4(u, v) = \frac{3}{8}X_4(u) - \frac{1}{2\,4}X_4^{(2)}\cos 2v + \frac{1}{1\,9\,2}X_4^{(4)}(u)\cos 4v \end{cases}$$

where

$$X_n^{(p)}(u) = (1-u^2)^{p/2}\frac{d^p X_n(u)}{du^p}, \quad p = 1, 2, \ldots n.$$

From (13), (14), and (15) we now get the expansion

$$(16) \quad . \quad . \quad \begin{cases} V' + \Omega = \frac{\omega^2}{3}r^2[1 - X_2(u)] \\ \quad + \frac{GM'}{R}\left\{1 + \frac{1}{2}q(1+\sigma) - \sigma\frac{r}{R}Y_1 + \sum_{n=2}^{\infty}\left(\frac{r}{R}\right)^n Y_n\right\}. \end{cases}$$

The terms for $n = 5, 6, \ldots$, while comparable with (8) or smaller, are to be neglected. The expression (16) should be introduced into the formula (1).

6. *Scalar Variables.*—In the study of the special case, where the disturbing component is treated as a point, there are only two gravity potentials, an inner, Φ, given by the formula

$$\Phi = U + V' + \Omega, \tag{17}$$

and an outer, Ψ, whose expression is

$$\Psi = V + V' + \Omega. \tag{18}$$

The two gravity-potentials satisfy the equations

$$\begin{cases} \Delta\Phi + 4\pi G\rho - 2\omega^2 = 0, \\ \Delta\Psi - 2\omega^2 = 0, \end{cases} \tag{19}$$

where ρ is given as a function of Φ by the relation (2). Further, certain conditions depending on the nature of the potential must be fulfilled.

By the linear substitution

$$\Phi - \tilde{\Phi} = G\frac{h}{l}\phi, \quad \Psi - \tilde{\Phi} = G\frac{h}{l}\psi \tag{20}$$

the scalar gravity potentials ϕ and ψ are introduced. The constant factor $G\frac{h}{l}$ is chosen so that

$$\phi = 1 \text{ at the centre of gravity of M.} \tag{21}$$

This implies that

$$G\frac{h}{l} = \Phi_c - \tilde{\Phi}, \tag{22}$$

where Φ_c is the value of the gravity-potential at the centre of gravity of M.

We have

$$\phi = 0 \text{ at the surface of M.} \tag{23}$$

Instead of r the scalar radius vector ξ is introduced by putting

$$r = l\xi. \tag{24}$$

The length l will be fixed shortly. Other independent variables are u and v.

Instead of the density ρ the scalar density χ is introduced by putting

$$\chi = \frac{b}{\kappa}\rho, \tag{25}$$

where the parameter κ has the value

$$\kappa = \frac{ab}{3.4^3}\left(\frac{m}{R}\right)^4 \frac{\beta^4}{1-\beta} G^3 \left(\frac{h}{l}\right)^3. \tag{26}$$

Between χ and ϕ there is, according to (2), a relation

$$(27) \quad \frac{\chi(1-\frac{3}{4}\kappa\chi)^3}{(1-\kappa\chi)^4}=\phi^3.$$

Now the length l is fixed by the condition

$$(28) \quad 4\pi h^{-1}l^3\kappa b^{-1}=1.$$

Further, a parameter λ is defined by the formula

$$(29) \quad \lambda=G^{-1}h^{-1}l^3 2\omega^2=\frac{\omega^2 b}{2\pi G\kappa}.$$

Then, the differential equations (19) take the form

$$(30) \quad \nabla\phi+\chi-\lambda=0, \qquad \nabla\psi-\lambda=0,$$

where ∇ is Laplace's differential symbol with ξ written instead of r, *i.e.*

$$\nabla=\frac{d^2}{d\xi^2}+\frac{2}{\xi}\frac{d}{d\xi}+\frac{1}{\xi^2}\frac{d}{du}\left[(1-u^2)\frac{d}{du}\right]+\frac{1}{\xi^2(1-u^2)}\frac{d^2}{dv^2}.$$

In addition to (21), the scalar gravity-potentials satisfy also the following conditions, because of the characteristic properties of the potential: we have

$$(31) \quad \phi-\psi=0, \qquad \dot\phi-\dot\psi=0 \text{ * at the surface of M;}$$

further, it follows from (16), (18), and (20) that ψ is of the form

$$(32) \quad \psi=\frac{\lambda}{6}\xi^2[1-X_2(u)]+\frac{M'}{h}\frac{l}{R}\left\{1+\tfrac{1}{2}q(1+\sigma)-\sigma\frac{\xi l}{R}Y_1+\sum_{n=2}^{\infty}\left(\frac{\xi l}{R}\right)^n Y_n\right\}$$
$$-\gamma+\frac{\delta}{\xi}+\frac{1}{\xi^3}P\left(\frac{1}{\xi}\right).$$

The constants γ and δ have the meaning [*cf.* (20), (22), and (24)]

$$(33) \quad \gamma=\frac{\tilde\Phi}{\Phi_c-\tilde\Phi},$$

$$(34) \quad \delta=\frac{M}{h}.$$

The radius of the star M in the spherical state when $R=\infty$ is denoted by $\bar\xi l$. The notation

$$(35) \quad \nu=\frac{\bar\xi l}{R}$$

is also introduced.

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* A dot means derivation with regard to ξ.

From (10) and (29) the parameter λ can be expressed in terms of ν by means of the relation

$$(36) \quad \lambda = 2\frac{M+M'}{h}(1+\sigma)\frac{l^3}{R^3} = 2\delta\left(1+\frac{M'}{M}\right)(1+\sigma)\left(\frac{\nu}{\xi}\right)^3.$$

Hence (32) may also be written

$$(37) \quad \begin{cases} \psi = \dfrac{M'}{M}\dfrac{\delta}{\xi}\nu\left\{1+\tfrac{1}{2}q(1+\sigma)-\sigma\nu\dfrac{\xi}{\bar{\xi}}Y_1\right\} \\ \quad +\tfrac{1}{6}\lambda\xi^2[1-X_2(u)]+\tfrac{1}{2}\lambda\xi^2\dfrac{q}{1+\sigma}\displaystyle\sum_{n=2}^{\infty}\left(\frac{\nu\xi}{\bar{\xi}}\right)^{n-2}Y_n \\ \quad -\gamma+\dfrac{\delta}{\bar{\xi}}+\dfrac{1}{\bar{\xi}^3}P\left(\dfrac{1}{\bar{\xi}}\right). \end{cases}$$

7. *Expansions according to Powers of* λ.—The required functions and constants are expanded according to ascending powers of λ as follows:—

$$(38) \quad \begin{cases} \phi = \phi_0+\lambda\phi_1+\lambda^2\phi_2+\ldots, \\ \psi = \psi_0+\lambda\psi_1+\lambda^2\psi_2+\ldots, \\ \chi = \chi_0+\lambda\chi_1+\lambda^2\chi_2+\ldots, \\ \gamma = \gamma_0+\lambda\gamma_1+\lambda^2\gamma_2+\ldots, \\ \delta = \delta_0+\lambda\delta_1+\lambda^2\delta_2+\ldots, \\ \sigma = \sigma_0+\lambda\sigma_1+\lambda^2\sigma_2+\ldots \end{cases}$$

The coefficients of these series are themselves polynomials of ν of at most the second degree. We know already that σ_0 disappears with ν. We will neglect the terms multiplied by λ^2.

From (27),

$$(39) \quad \frac{\chi_0(1-\frac{3}{4}\kappa\chi_0)^3}{(1-\kappa\chi_0)^4} = \phi_0^3.$$

Further,

$$(40) \quad \chi_1 = \frac{d\chi_0}{d\phi_0}\phi_1 = 3\chi_0^{2/3}(1-\kappa\chi_0)^{7/3}\cdot\phi_1.$$

The differential equations (30) now split up into the following pairs of equations:—

$$(41) \quad \nabla\phi_0+\chi_0 = 0, \qquad \nabla\psi_0 = 0,$$

$$(42) \quad \nabla\phi_1+\chi_1-1 = 0, \qquad \nabla\psi_1-1 = 0,$$

. . . . . .

When λ (and ν) vanish we come back to the system of equations (41), already studied in II. The initial values of ϕ_0 are

$$\phi_0 = 1, \qquad \phi_0' = 0, \text{ when } \xi = 0.$$

ϕ_0, which is a function of ξ alone, is thus completely defined. Its first zero $\bar{\xi}$ is a function of κ. In what follows, a line over a function means its value or expression when $\xi = \bar{\xi}$. For perfectly gaseous stars the function $\phi_0(\xi)$ and its zero $\bar{\xi}$ have been numerically calculated in my paper, II. pp. 701 and 690.

The general solution of the second equation of (41) is of the form

$$\psi_0 = A + \frac{B}{\xi},$$

since ψ_0 is independent of u and v. The constants A and B are determined by the conditions

$$\bar{\psi}_0 = \bar{\phi}_0 = 0,$$
$$\bar{\psi}_0 = \bar{\phi}_0.$$

Hence it follows that

$$B = -\bar{\xi}^2\bar{\phi}_0, \qquad A = \bar{\xi}\bar{\phi}_0.$$

Thus we have

$$(43) \quad \psi_0 = \bar{\xi}\bar{\phi}_0 - \frac{\bar{\xi}^2\bar{\phi}_0}{\xi}.$$

A comparison with the expansion (37) shows that

$$(44) \quad \delta_0 = -\bar{\xi}^2\bar{\phi}_0,$$

$$(45) \quad \gamma_0 = -\bar{\xi}\bar{\phi}_0\left\{1 + \nu\frac{M'}{M}\left[1 + \tfrac{1}{2}q(1+\sigma_0)\right]\right\}.$$

I will proceed to look for the conditions which ϕ_1 and ψ_1 must satisfy.

I take as my starting-point the equation (23) of the surface, which may be written

$$\phi_0 + \lambda\phi_1 + \ldots = 0,$$

i.e. after expansion according to powers of $\xi - \bar{\xi}$

$$\bar{\phi}_0(\xi - \bar{\xi}) + \ldots + \lambda\bar{\phi}_1 + \ldots = 0.$$

At the surface we thus get ξ as a function of u and v under the form

$$(46) \quad \xi = \bar{\xi} + \lambda\xi_1 + \lambda^2\xi_2 + \ldots.$$

Obviously

$$(47) \quad \xi_1 = -\frac{\bar{\phi}_1}{\bar{\phi}_0}.$$

The conditions (31) now become after expansion

$$\bar{\phi}_0 - \bar{\psi}_0 + \lambda[(\bar{\phi}_0 - \bar{\psi}_0)\xi_1 + \bar{\phi}_1 - \bar{\psi}_1] + \ldots = 0,$$
$$\bar{\phi}_0 - \bar{\psi}_0 + \lambda[(\bar{\phi}_0 - \bar{\psi}_0)\xi_1 + \bar{\phi}_1 - \bar{\psi}_1] + \ldots = 0.$$

The terms independent of λ drop out. Further $\bar{\phi}_0 - \bar{\psi}_0 = 0$ because of (41), since $\bar{\chi}_0 = 0$. Thus the functions ϕ_1 and ψ_1 satisfy the conditions

$$(48) \quad . \quad . \quad . \qquad \bar{\phi}_1 - \bar{\psi}_1 = 0, \quad \dot{\phi}_1 - \dot{\bar{\psi}}_1 = 0.$$

A third condition,

$$(49) \quad . \quad . \quad . \quad . \quad . \qquad \phi_1 = 0, \text{ when } \dot{\xi} = 0,$$

follows from (21), which must be satisfied identically for all values of ν.

From (37) it follows as a fourth condition that ψ_1 must be of the form

$$(50) \quad . \quad \begin{cases} \psi_1 = \dfrac{M'}{M}\dfrac{\nu}{\xi}\{[1 + \tfrac{1}{2}q(1+\sigma_0)]\delta_1 + \tfrac{1}{2}q\sigma_1\delta_0\} + \tfrac{1}{6}\xi^2[1 - X_2(u)] \\ \quad - \dfrac{M'}{M}\delta_0\dfrac{\nu^2}{\xi^2}\left(\dfrac{\sigma_0}{\lambda} + \sigma_1\right)\xi Y_1 + \dfrac{1}{2}\dfrac{q}{1+\sigma_0}\xi^2\sum_{n=2}^{4}\left(\dfrac{\nu\xi}{\xi}\right)^{n-2} Y_n \\ \quad - \gamma_1 + \dfrac{\delta_1}{\xi} + \dfrac{1}{\xi^3} P_1\left(\dfrac{1}{\xi}\right). \end{cases}$$

8. *Expansions according to Spherical Harmonics.*—I pass on to the integration of the system (42). I introduce the notation

$$(51) \quad . \quad . \quad . \qquad E(\xi\,;\ \kappa) = \frac{d\chi_0}{d\phi_0} = 3\chi_0^{2/3}(1 - \kappa\chi_0)^{7/3}.$$

From (40) it follows then that (42) may be written

$$(52) \quad . \quad . \quad . \qquad \nabla\phi_1 + E\,.\,\phi_1 = 1, \quad \nabla\psi_1 = 1.$$

The function $E(\xi\,;\ \kappa)$ may be considered numerically known from $\xi = 0$ to $\xi = \bar{\xi}(\kappa)$. In the vicinity of $\xi = 0$ E may be expanded in a series of powers of ξ^2. See also II. p. 690.

The functions ϕ_1 and ψ_1 are developed according to spherical harmonics $Z(u, v)$ in the form

$$(53) \quad . \quad . \quad . \qquad \begin{cases} \phi_1 = \phi_{1.0}(\xi) + \Sigma P(\xi) Z(u, v), \\ \psi_1 = \psi_{1.0}(\xi) + \Sigma Q(\xi) Z(u, v). \end{cases}$$

These developments are introduced into the differential equations (52). From the theory of spherical harmonics we get the important formula

$$\nabla[F(\xi)Z(u, v)] = Z(u, v)\,.\,D_n F(\xi),$$

where n is the order of the spherical function Z, and D_n denotes the symbol

$$D_n = \frac{d^2}{d\xi^2} + \frac{2}{\xi}\frac{d}{d\xi} - \frac{n(n+1)}{\xi^2}.$$

The system of partial differential equations (52) thus splits up into an infinite number of pairs of ordinary differential equations, one pair for each spherical harmonic Z:

$$(54) \quad . \quad . \quad . \qquad D_0\phi_{1.0} + E\,.\,\phi_{1.0} = 1, \quad D_0\psi_{1.0} = 1;$$

$$(55) \quad . \quad . \qquad D_n P + EP = 0, \quad D_n Q = 0, \quad n = 1, 2, 3, \ldots$$

It is easy to find the form of the function $\psi_{1.0}$. The general solution of (54_2) is of the form

$$(56) \quad \psi_{1.0} = \tfrac{1}{6}\xi^2 + A + \frac{B}{\xi},$$

where A and B are undetermined constants. Because of (50), these constants depend on γ_1 and δ_1. We do not need the expression of A. For B we find

$$(56') \quad B = \delta_1.$$

It is also easy to find the expression for Q. The general solution of (55_2) is

$$(57) \quad Q = A\xi^n + \frac{B}{\xi^{n+1}},$$

where A and B are undetermined constants.

Let us first study the coefficients Q of the spherical harmonics of the first order in the series (53_2). In that case $n = 1$. From comparison with (50) it appears that

$$(58) \quad A = -\frac{M'}{M}\delta_0 \frac{\nu^2}{\bar{\xi}^2}\left(\frac{\sigma_0}{\lambda} + \sigma_1\right), \quad B = 0, \text{ if } Z \equiv Y_1,$$

and that

$$(59) \quad A = B = 0,$$

if Z is a spherical harmonic of the first order other than Y_1.

Let us next suppose that Z is of a higher order than 1. In the expression (57) for the corresponding Q we have

$$(60) \quad \begin{cases} A = -\frac{1}{6}, & \text{if } Z = X_2(u); \\ A = \frac{1}{2}\,\frac{q}{1+\sigma_0}\left(\frac{\nu}{\bar{\xi}}\right)^{n-2}, & \text{if } Z = Y_n, \quad n = 2, 3, 4; \\ A = 0 \text{ in other cases.} \end{cases}$$

In all these cases $(n > 1)$ the constant B in (57) is still undetermined.

I pass on to the equation (54_1). It has a perfectly definite particular solution

$$(61) \quad \phi_{1.0} = \frac{\xi^2}{6}[1 + k_1\xi^2 + k_2\xi^4 + k_3\xi^6 + \ldots].$$

The expressions for the coefficients k_s have already been given [II. (69)]. The homogeneous equation has a particular solution which is $= 1$, when $\xi = 0$; and another particular solution which is $= \infty$, as ξ^{-1} when $\xi = 0$. The general solution of (54_1) is the sum of the particular solution (61) and a linear combination with constant coefficients of the above solutions of the homogeneous equation. But it follows from the condition

$$\phi_{1.0} = 0, \text{ when } \xi = 0,$$

which is a consequence of the condition (49), that these constant coefficients must be put = 0. Thus $\phi_{1.0}$ is given by (61). This function can be calculated numerically throughout the whole interval $0<\xi<\bar{\xi}$. The calculations have already been made for perfectly gaseous stars in II. pp. 701 and 694.

I proceed to the equation (55_1). It has a perfectly definite particular solution, depending only on the parameter κ and the index n, of the form

$$(62) \quad . \quad . \quad f_n = \xi^n[1 + L_{n,1}\xi^2 + L_{n,2}\xi^4 + L_{n,3}\xi^6 + \ldots],$$

and another particular solution which, when $\xi = 0$, is $= \infty$ as ξ^{-n-1}. The general solution is a linear combination of these two. But since, from (49), $P = 0$ when $\xi = 0$, the infinite solution drops out and

$$(63) \quad . \quad . \quad . \quad . \quad . \quad P = Cf_n,$$

where C are constants as yet undetermined.

The coefficients of (62) have been found to be

$$(64) \quad . \quad \begin{cases} L_{n,1} = -\dfrac{3}{2(2n+3)}\dfrac{(1-\alpha)^5}{(1-\frac{3}{4}\alpha)^2}, \\ L_{n,2} = \dfrac{1}{8(2n+5)}\dfrac{(1-\alpha)^9}{(1-\frac{3}{4}\alpha)^4}\left\{(2-9\alpha)+\dfrac{9}{2n+3}(1-\alpha)\right\}, \\ L_{n,3} = -\dfrac{1}{6(2n+7)}\dfrac{(1-\alpha)^{13}}{(1-\frac{3}{4}\alpha)^6}\left\{\dfrac{1}{120}(28-399\alpha+756\alpha^2)\right. \\ \qquad +\dfrac{3}{4}\dfrac{1}{2n+3}(1-\alpha)(2-9\alpha)+\dfrac{3}{8(2n+5)}(1-\alpha)(2-9\alpha) \\ \qquad \left.+\dfrac{27}{8(2n+3)(2n+5)}(1-\alpha)^2\right\}, \end{cases}$$

where α and κ are connected by the relation

$$(65) \quad . \quad . \quad . \quad . \quad \frac{\alpha(1-\frac{3}{4}\alpha)^3}{(1-\alpha)^4} = \kappa.$$

For $\kappa = 0$, $\alpha = 0$ (perfectly gaseous stars) the expressions for the coefficients are

$$(66) \quad . \quad . \quad \begin{cases} L_{n,1} = -\dfrac{3}{2(2n+3)}, \\ L_{n,2} = +\dfrac{4n+15}{8(2n+3)(2n+5)}, \\ L_{n,3} = -\dfrac{112n^2+988n+1995}{720(2n+3)(2n+5)(2n+7)}. \end{cases}$$

The functions $f_n(\xi\,;\kappa)$ can be numerically calculated within the region $0 \leq \xi \leq \bar{\xi}$ for every given value of κ.

The calculation of f_2 for perfectly gaseous stars has already been

made in the paper II. p. 701. The numerical values of f_3 and f_4 for perfect gas stars are to be found in the same place.

The functions Q and P, given by (57) and (63), must satisfy two more conditions

$$(67) \quad . \quad . \quad . \quad . \quad P - \bar{Q} = 0, \quad \bar{P} - \bar{Q} = 0,$$

which follow from (48). These conditions fix the undetermined constants.

If Z is a spherical harmonic of the first order other than Y_1, it follows from (57) and (59) that $Q \equiv 0$, and so, according to (63) and (67), that also $P \equiv 0$.

If $Z \equiv Y_1$, then it follows from (57), (58), and (63) that the conditions (67) are of the form

$$A\bar{\xi} - C\bar{f}_1 = 0,$$
$$A - C\bar{f}_1 = 0.$$

The determinant of this system is, as will now be shown, $\neq 0$. In fact, we get by derivation from (41) the relation

$$D_1\phi_0 + E \cdot \phi_0 = 0.$$

Hence
$$f_1 = K\phi_0,$$

where K is a certain constant $\neq 0$. Thus

$$\bar{\xi}\bar{f}_1 - \bar{f}_1 = K[\bar{\xi}\bar{\phi}_0 - \bar{\phi}_0] = -3K\bar{\phi}_0 \neq 0.$$

Thus if $Z \equiv Y_1$, then not only $B = 0$ [*cf.* (58_2)], but also $A = 0$ and $C = 0$. From $A = 0$ [*cf.* (58_1)] it follows that

$$(68) \quad . \quad . \quad . \quad . \quad . \quad \sigma_0 = 0,$$

$$(69) \quad . \quad . \quad . \quad . \quad . \quad \sigma_1 = 0, \text{ when } \nu = 0.$$

Thus no spherical harmonics of the first order appear in the developments (53) for ϕ_1 and ψ_1.

I proceed to determine the coefficients of the spherical harmonics of second and higher orders in (53). From the expressions (57) and (63), the conditions (67) may be written

$$C\bar{f}_n - B\bar{\xi}^{-n-1} = A\bar{\xi}^n,$$
$$C\bar{f}_n + (n+1)B\bar{\xi}^{-n-2} = nA\bar{\xi}^{n-1}. \qquad n = 2, 3, 4, \ldots,$$

The determinant of this system,

$$(70) \quad . \quad . \quad . \quad . \quad \bar{\Delta}_n = \bar{\xi}\bar{f}_n + (n+1)\bar{f}_n,$$

is in point of fact not identically $= 0$ for all values of κ if $n \geq 2$. [On the contrary, $\bar{\Delta}_1$ is $\equiv 0$.] For $\kappa = 0$, it is found that $\bar{\Delta}_n \neq 0$ $(n \geq 2)$. We therefore get the expressions

$$(71) \quad . \quad . \quad . \quad . \quad C = \frac{(2n+1)A}{\bar{\Delta}_n}\bar{\xi}^n,$$

$$(72) \quad . \quad . \quad . \quad . \quad B = \frac{n\bar{f}_n - \bar{\xi}\bar{f}_n}{\bar{\Delta}_n} A \cdot \bar{\xi}^{2n+1}.$$

If $Z \neq X_2(u)$ and $\neq Y_n(u, v)$ $(n = 2, 3, 4)$, then, according to (60), $A = 0$. Thus, in consequence of (71) and (72), $B = C = 0$ also. Such spherical harmonics do not then appear in the developments (53) for ϕ_1 and ψ_1.

If, however, $Z = X_2(u)$ or $= Y_n(u, v)$ $(n = 2, 3, 4)$, then A is given by (60) and C and B by (71) and (72).

We are now in a position to give the complete expression for ϕ_1:

$$(73)\quad \phi_1 = \phi_{1.0}(\xi) - \frac{5}{6}\frac{\bar{\xi}^2}{\bar{\Delta}_2} f_2(\xi) \cdot X_2(u) + \tfrac{1}{2} q \bar{\xi}^2 \sum_{n=2}^{4} \frac{2n+1}{\bar{\Delta}_n} \nu^{n-2} f_n(\xi) Y_n(u, v).$$

I return to the formulæ (56), (56′), and (61). From (48) the functions $\phi_{1.0}$ and $\psi_{1.0}$ must satisfy the conditions

$$(74)\quad \ldots \quad \bar{\phi}_{1.0} - \bar{\psi}_{1.0} = 0, \quad \dot{\bar{\phi}}_{1.0} - \dot{\bar{\psi}}_{1.0} = 0.$$

These conditions fix A and B in (56) and so also γ_1 and δ_1. The second condition (74) leads to the expression

$$(75)\quad \ldots \quad \delta_1 = \tfrac{1}{3}\bar{\xi}^3 - \bar{\xi}^2 \cdot \dot{\bar{\phi}}_{1.0}.$$

We will pass over the expression for γ_1.

I proceed to establish the definitive expression for the inner scalar gravity-potential ϕ. In the development (38_1) we introduce for λ, according to (36), (44), and (68), the expression

$$(76)\quad \ldots \quad \lambda = -\frac{2\bar{\phi}_0}{\bar{\xi}}\left(1 + \frac{M'}{M}\right)\nu^3.$$

Finally, with reference to the notation (11) and the formula (73) we get

$$(77)\quad \begin{cases} \phi = \phi_0(\xi) - 2\dfrac{\bar{\phi}_0}{\bar{\xi}}\left(1 + \dfrac{M'}{M}\right)\nu^3\left\{\phi_{1.0}(\xi) - \dfrac{5}{6}\dfrac{\bar{\xi}^2}{\bar{\Delta}_2} f_2(\xi) X_2(u)\right\} \\ \qquad - \bar{\xi}\bar{\phi}_0 \dfrac{M'}{M}\nu^3 \displaystyle\sum_{n=2}^{4} \frac{2n+1}{\bar{\Delta}_n}\nu^{n-2} f_n(\xi) Y_n(u, v). \end{cases}$$

9. *External Shape.*—The equation of the surface, $\phi = 0$, after solution gives ξ as a function of u and v at the surface. We introduce the notation

$$(78)\quad \ldots \quad \bar{\zeta}_n = \bar{\xi}\frac{\bar{f}_n}{\dot{\bar{f}}_n}.$$

$\bar{\zeta}_n$ is a constant, dependent only on κ. With this notation we have

$$\frac{\bar{f}_n}{\bar{\Delta}_n} = \frac{1}{\bar{\zeta}_n + n + 1}.$$

Thus we find at the surface, since $\bar{\phi}_0 = 0$,

$$(79)\quad \frac{\xi - \bar{\xi}}{\bar{\xi}} = 2\left(1 + \frac{M'}{M}\right)\nu^3\left\{\frac{\bar{\phi}_{1.0}}{\bar{\xi}^2} - \frac{5}{6}\frac{1}{\bar{\zeta}_2 + 3} X_2(u)\right\} + \frac{M'}{M}\nu^3 \sum_{n=2}^{4} \frac{2n+1}{\bar{\zeta}_n + n + 1}\nu^{n-2} Y_n(u, v).$$

The volume N of the star is obtained with sufficient approximation from the formula

$$(80)\quad \left\{\begin{aligned} N &= \frac{4\pi}{3} l^3 \bar{\xi}^3 + l^3 \int_0^{2\pi}\int_{-1}^{+1} (\xi - \bar{\xi})\bar{\xi}^2 du dv \\ &= \frac{4\pi}{3} l^3 \bar{\xi}^3 \left\{ 1 + 6\left(1 + \frac{M'}{M}\right)\nu^3 \frac{\bar{\phi}_{1.0}}{\bar{\xi}^2} + \ldots \right\} \\ &= \frac{4\pi}{3} l^3 \bar{\xi}^3 \left\{ 1 + \lambda \frac{3\bar{\phi}_{1.0}}{-\bar{\xi}\bar{\phi}_0} + \ldots \right\}. \end{aligned}\right.$$

We define the mean radius r_1 of the star as the radius of the sphere with the same volume N as the star. Thus

$$(81)\quad \left\{\begin{aligned} r_1 &= l\bar{\xi}\left\{ 1 + 2\left(1 + \frac{M'}{M}\right)\nu^3 \frac{\bar{\phi}_{1.0}}{\bar{\xi}^2} + \ldots \right\} \\ &= l\bar{\xi}\left\{ 1 + \lambda \frac{\bar{\phi}_{1.0}}{-\bar{\xi}\bar{\phi}_0} + \ldots \right\}. \end{aligned}\right.$$

With this introduced, the equation of the surface (79) may also be written in the form

$$(82)\quad \frac{r - r_1}{r_1} = -\frac{5}{3}\left(1 + \frac{M'}{M}\right)\frac{\nu^3}{\bar{\zeta}_2 + 3} X_2(u) + \frac{M'}{M}\sum_{n=2}^{4} (2n+1)\frac{\nu^{n+1}}{\bar{\zeta}_n + n + 1} X_n(u_1),$$

where

$$(82')\quad u_1 = \frac{x}{r}.$$

After introducing the mean radius r_1 we may treat the parameter ν as the quotient between r_1 and R

$$(83)\quad \nu = \frac{r_1}{R}.$$

Through (82) r is expressed as a function of its direction-cosines u and u_1 with the axis of z and of x. Evidently u and u_1 are limited by the condition

$$u^2 + u_1^2 \leq 1.$$

The surface (82) of the star is symmetrical with regard to the equator and the principal meridian (passing through the component), but it is unsymmetrical as regards the plan of $yz(u_1 = 0)$.

The radius vector has a principal maximum

$$r_1\left\{ 1 + \tfrac{5}{6}\left(1 + 7\frac{M'}{M}\right)\frac{\nu^3}{\bar{\zeta}_2 + 3} + 7\frac{M'}{M}\frac{\nu^4}{\bar{\zeta}_3 + 4} + 9\frac{M'}{M}\frac{\nu^5}{\bar{\zeta}_4 + 5} \right\}$$

towards M', and a secondary maximum

$$r_1\left\{ 1 + \tfrac{5}{6}\left(1 + 7\frac{M'}{M}\right)\frac{\nu^3}{\bar{\zeta}_2 + 3} - 7\frac{M'}{M}\frac{\nu^4}{\bar{\zeta}_3 + 4} + 9\frac{M'}{M}\frac{\nu^5}{\bar{\zeta}_4 + 5} \right\}$$

in the opposite direction. The principal minima of r are situated in the principal meridian making the angle

$$\frac{21}{10}\frac{M}{M+4M'}\frac{\bar{\zeta}_2+3}{\bar{\zeta}_3+4}\nu+(\nu^3)$$

with the axis of rotation towards M′. Finally, r has two minimaxima * in the equator forming an angle

$$\frac{7}{10}\frac{\bar{\zeta}_2+3}{\bar{\zeta}_3+4}\nu+(\nu^3)$$

with the axis of y, also towards M′.

The diameter of the surface through the centre of gravity has its principal maximum

$$2r_1\left\{1+\tfrac{5}{6}\left(1+7\frac{M'}{M}\right)\frac{\nu^3}{\bar{\zeta}_2+3}+9\frac{M'}{M}\frac{\nu^5}{\bar{\zeta}_4+5}\right\}$$

along the line MM′, and its principal minimum

$$2r_1\left\{1-\tfrac{5}{3}\left(1+\frac{5}{2}\frac{M'}{M}\right)\frac{\nu^3}{\bar{\zeta}_2+3}+\frac{27}{8}\frac{M'}{M}\frac{\nu^5}{\bar{\zeta}_4+5}\right\}$$

along the axis of rotation. Along the axis of y the diameter has the minimaximum

$$2r_1\left\{1+\tfrac{5}{6}\left(1-2\frac{M'}{M}\right)\frac{\nu^3}{\bar{\zeta}_2+3}+\frac{27}{8}\frac{M'}{M}\frac{\nu^5}{\bar{\zeta}_4+5}\right\}.$$

Numerical calculation for $\kappa=0$, *i.e.* for perfectly gaseous stars, has given the following values for the quantities $\bar{\zeta}_n$:

$$(84)\quad \begin{cases}\bar{\zeta}_2=+1{\cdot}8597,\\ \bar{\zeta}_3=+2{\cdot}9486,\\ \bar{\zeta}_4=+3{\cdot}9747.\end{cases}$$

10. *Mass and Mean Density.*—The expression for the mass M is, according to (34),

$$M=h\delta=h(\delta_0+\lambda\delta_1+\ldots).$$

Using the formulæ (44), (75), and (76) we get

$$(85)\quad \begin{cases} M=-h\bar{\xi}^2\phi_0^I\left\{1+\lambda\dfrac{\frac{1}{3}\bar{\xi}^2-\bar{\xi}\phi_{1\cdot0}^I}{-\bar{\xi}\phi_0^I}+\ldots\right\}\\ \quad=-h\bar{\xi}^2\phi_0^I\left\{1+2\left(1+\dfrac{M'}{M}\right)\nu^3\left(\frac{1}{3}-\dfrac{\phi_{1\cdot0}^I}{\bar{\xi}}\right)+\ldots\right\}.\end{cases}$$

The mean density is from (85), (80), and (28)

$$(86)\quad \begin{cases}\rho_m=\dfrac{M}{\bar{N}}=-\dfrac{3\phi_0^I}{\bar{\xi}}\dfrac{\kappa}{b}\left\{1-\dfrac{\lambda}{-\bar{\xi}\phi_0^I}(\bar{\xi}\phi_{1\cdot0}^I+3\bar{\phi}_{1\cdot0}-\tfrac{1}{3}\bar{\xi}^2)+\ldots\right\}\\ \quad=-\dfrac{3\phi_0^I}{\bar{\xi}}\dfrac{\kappa}{b}\left\{1-2\left(1+\dfrac{M'}{M}\right)\nu^3\left(\dfrac{\phi_{1\cdot0}^I}{\bar{\xi}}+\dfrac{3\bar{\phi}_{1\cdot0}}{\bar{\xi}^2}-\frac{1}{3}\right)+\ldots\right\}.\end{cases}$$

* This word indicates a relationship such as holds in the radius vector of an ellipsoid at the end of axis b, where $a>b>c$.

The formulæ (80), (85), and (86) have exactly the same form as in the theory for a single rotating star.

From (86), (29), and (76) we get, omitting ν^6, the relations

$$(87)\quad \frac{\omega^2}{2\pi G\rho_m} = -\frac{\bar{\xi}}{3\bar{\phi}_0}\frac{\omega^2 b}{2\pi G\kappa} + \ldots = -\frac{\bar{\xi}}{3\bar{\phi}_0}\lambda + \ldots = \tfrac{2}{3}\left(1+\frac{M'}{M}\right)\nu^3 + \ldots .$$

For perfectly gaseous stars we have $\kappa = 0$, $\chi = \phi^3$; thus $\chi = 1$ at the centre of gravity; thus $\frac{\kappa}{b} = \rho_c$, where ρ_c means the central density.

Numerical calculation has given for a perfectly gaseous star, using (86),

$$(88)\quad \frac{\rho_m}{\rho_c} = 0{\cdot}018455\left\{1 - 0{\cdot}6610\left(1+\frac{M'}{M}\right)\nu^3 + \ldots\right\}.$$

11. *Variation of Gravity on the Surface.*—The following expression applies to the gravity, if λ^2 is neglected:

$$\frac{d\Phi}{dn_i} = -\frac{d\Phi}{dr} + \ldots = -\frac{Gh}{l^2}\phi + \ldots$$

$$= \frac{Gh}{l^2}\{-\phi_0 - \lambda\phi_1 + \ldots\}.$$

At the surface, according to (46) and (47),

$$-\phi_0 = -\bar{\phi}_0 + \bar{\phi}_0\lambda\frac{\bar{\phi}_1}{\bar{\phi}_0} + \ldots = -\bar{\phi}_0 - \lambda\frac{2\bar{\phi}_1}{\bar{\xi}} + \ldots$$

Thus the gravity on the surface is

$$\frac{d\tilde{\Phi}}{dn_i} = -\bar{\phi}_0\frac{Gh}{l^2}\left\{1 + \frac{\lambda}{\bar{\xi}\bar{\phi}_0}(\bar{\xi}\bar{\phi}_1 + 2\bar{\phi}_1) + \ldots\right\}.$$

From (81) and (85) is derived

$$\frac{M}{r_1^2} = -\bar{\phi}_0\frac{h}{l^2}\left\{1 + \frac{\lambda}{-\bar{\xi}\bar{\phi}_0}(\tfrac{1}{3}\bar{\xi}^2 - \bar{\xi}\bar{\phi}_{1.0} - 2\bar{\phi}_{1.0}) + \ldots\right\}.$$

Of these formulæ the former is divided by the latter. Instead of λ its expression (76) is used. Further, in accordance with (73) and (78),

$$\bar{\xi}(\bar{\phi}_1 - \bar{\phi}_{1.0}) + 2(\bar{\phi}_1 - \bar{\phi}_{1.0}) = -\tfrac{5}{6}\bar{\xi}^2\frac{\bar{\zeta}_2 + 2}{\bar{\zeta}_2 + 3}X_2(u)$$

$$+ \frac{q}{2}\bar{\xi}^2\sum_{n=2}^{4}(2n+1)\frac{\bar{\zeta}_n + 2}{\bar{\zeta}_n + n + 1}\nu^{n-2}X_n(u_1)$$

is introduced. We thus get finally for gravity at the surface

$$(89)\quad \left\{\begin{aligned}\frac{d\tilde{\Phi}}{dn_i}=\frac{GM}{r_1^2}\Big\{1-\tfrac{2}{3}\Big(1+\frac{M'}{M}\Big)\nu^3\Big[1-\frac{5}{2}\frac{\zeta_2+2}{\bar{\zeta}_2+3}X_2(u)\Big]\\ -\frac{M'}{M}\nu^3\sum_{n=2}^{4}(2n+1)\frac{\zeta_n+2}{\bar{\zeta}_n+n+1}\nu^{n-2}X_n(u_1)+\ldots\Big\}.\end{aligned}\right.$$

Thus gravity has on the surface a principal minimum towards the component and a secondary minimum in the opposite direction. The maximum of gravity lies in the principal meridian near the pole, where the radius vector makes the angle with the axis of rotation

$$(89')\quad \ldots\quad \frac{21}{10}\frac{M'}{M+4M'}\frac{\bar{\zeta}_2+3}{\bar{\zeta}_2+2}\frac{\bar{\zeta}_3+2}{\bar{\zeta}_3+4}\cdot\nu+(\nu^3)$$

towards the side of the component.

12. *Surface Brightness and Effective Temperature.*—We have shown in I. that the surface brightness is given generally by the formula

$$[\text{I. (36)}]\quad \ldots\quad \tilde{F}=\frac{c}{C}\frac{d\tilde{\Phi}}{dn_i}.$$

Here c is the velocity of light; C is a constant within the star.

If the absorption-coefficient k follows the law

$$[\text{I. (43)}]\quad \ldots\quad k=Kp T^{-4},$$

where p is the gas pressure and T the temperature, while K is a constant, then C depends on β by the formula

$$[\text{I. (44)}]\quad \ldots\quad C=\frac{a}{3}K\frac{\beta}{(1-\beta)^2}.$$

Hence the surface brightness is

$$(90)\quad \ldots\quad \tilde{F}=\frac{3c}{aK}\frac{(1-\beta)^2}{\beta}\frac{d\tilde{\Phi}}{dn_i}.$$

The effective temperature T_e is connected with the surface brightness by the formula

$$(91)\quad \ldots\quad T_e^4=\frac{4}{ac}\tilde{F}=\frac{12}{a^2K}\frac{(1-\beta)^2}{\beta}\frac{d\tilde{\Phi}}{dn_i}.$$

Thus the effective temperature has, just as the gravity, an absolute minimum towards the component, a relative minimum in the opposite direction turned away from the component, and an absolute maximum in the vicinity of the poles.

The constant β is obtained from the equation

$$(92)\quad \ldots\quad \frac{1-\beta}{\beta^4}=\frac{\pi a}{48}\Big(\frac{m}{R}\Big)^4 G^3\Big(\frac{M}{\delta}\Big)^2,$$

which is easily derived from (26), (28), and (34).

13. *Density, Temperature, and Radiation in the Interior of the Star. Fundamental Parameters.*—The scalar gravity-potential ϕ has been fixed above as a function of the variables $\xi = \frac{r}{l}$, u and v, and the parameters κ, ν, and $\frac{M'}{M}$. As fundamental parameters we may choose

(93) . . . . r_1, R, M, and M′.

As for ν, it is expressed by r_1 and R according to (83). The parameter κ is obtained in terms of the fundamental parameters by the equation (86) when ρ_m has been expressed in terms of M and r_1. Thus we get the equation

$$(94) \quad \frac{M}{r_1^3} = -4\pi \frac{\bar{\phi}_0}{\bar{\xi}} \frac{\kappa}{b} \left\{ 1 - 2\left(1 + \frac{M'}{M}\right)\nu^3 \left(\frac{\bar{\phi}_{1,0}}{\bar{\xi}} + \frac{3\bar{\phi}_{1,0}}{\bar{\xi}^2} - \tfrac{1}{3} \right) + \ldots \right\}.$$

Finally, the parameter l may be replaced by r_1 according to (81).

The constant β, determined by (92), depends obviously only on the fundamental parameters (if m is fixed).

From (25) and (27) the density ρ is obtained by the formula

$$(95) \quad \ldots \quad \frac{b\rho(1 - \frac{3}{4}b\rho)^3}{(1 - b\rho)^4} = \kappa\phi^3,$$

and thus depends on r, u, v and the fundamental parameters.

The temperature T satisfies the equation of the fourth degree [*cf.* I. (46)]

$$(96) \quad \ldots \quad \frac{a}{3}\frac{b}{1-\beta}T^4 + \frac{R}{m}\frac{1}{\beta}T = \left(G\frac{h}{l}\phi\right)^3.$$

Gas pressure p and radiation pressure ϖ are obtained from the formulæ [*cf.* I. (42), (25)]

$$(97) \quad \ldots \quad p = \frac{a}{3}\frac{\beta}{1-\beta}T^4, \quad \varpi = \frac{a}{3}T^4.$$

In accordance with the general theory [I. p. 669] radiation intensity is fully determined at every point and in every direction by density, temperature, and absorption coefficient.

Hence the state of the star M depends in every respect on the four fundamental parameters (93) above.

We have so far only considered one of the components. The state of the other component depends on the parameters r_1', R, M′, and M.

The condition of the double-star system is therefore completely fixed by five parameters

$$r_1,\ r_1',\ R,\ M,\ M'$$

(the two mean radii, the distance between the centres of gravity, and the two masses).

Electrical State of a Star. By S. Rosseland.

(*Communicated by Prof. A. S. Eddington.*)

1. In Eddington's theory of gaseous stars in hydrostatic equilibrium the star-building material is supposed to follow the ideal gas laws. From laboratory experiments this is known to be very nearly true for all gases within certain intervals of temperature and density, and in particular that gases under conditions like that of atmospheric air will be nearly perfect. The physical condition underlying this fact is the circumstance that the atoms will be electrically neutral and exert no forces on each other except at close collisions. In the stellar case things are different, because the atoms are strongly ionised, and the forces between the free particles in the gas will, except for very close collisions, follow the inverse square law, and thus be of an altogether higher order of magnitude than the residual forces acting between neutral atoms. It may therefore be of interest to undertake a more detailed investigation into the possible effect of the electrical forces on the pressure in a star, and this will be done in the present paper.

For the purpose of calculating the additional pressure which is due to electrical action it is convenient to consider directly the forces acting on an ultimate particle in the gas. There is first the possibility to be considered of the star possessing a resultant electric charge distributed throughout its volume. This charge will give rise to a force on the specified particle in the same way as is the case for gravitational attraction. Second, there must exist a force on the particle due to an unsymmetrical electric polarisation of the medium surrounding the particle. Thirdly, at very close collisions between positive ions and other particles the effect can no more be calculated on the basis of the inverse square law, and this fact will give rise to effects corresponding in a certain sense to the van der Waals correction in volume in the ordinary equation of state. It has already been anticipated by Eddington,* by elementary considerations, that the electrical pressure as it arises from forces between the particles following the inverse square law must be proportional to gas pressure. Therefore the conclusions regarding stellar evolution, which were obtained by assuming the ideal gas laws to be valid, would be expected to remain unaltered when taking account of the electrical pressure, regardless of the absolute value of the latter.

The following discussion agrees with this view for the case that the generation of energy in a star takes place in a manner conforming to Eddington's original theory. It tends, moreover, to prove that the absolute value of the pressure will be so small compared with that of gas pressure, that for the theory of the constitution of a gaseous star it is sufficient to assume the material to be in a perfect state as long as the density is so small that the dimensions of the positive ions are small compared with the mean distance between the free particles. As emphasised by Eddington, this conclusion has far-reaching importance for the theory of stellar evolution.

* *M.N.*, 84, 308, 1924.

2. *Net Electrical Charge of a Star.*—It has been recognised by several authors that a star constructed of strongly ionised matter would be expected to carry a resultant positive electric volume charge—it being tacitly assumed that the conditions are not seriously disturbed by radio-active emission of charged particles from the surface of the star. In order to deal with the principal point of the problem it is sufficient to consider the case of the distribution of positive and negative ions in a star which is in isothermal equilibrium, in which case the problem can be simply considered on the basis of Boltzmann's principle. When the case of varying temperature is also considered the conditions will be complicated with the effect of the so-called "thermal diffusion." The latter effects may, however, be expected to be negligible for the present case, as the variation in temperature within a star on Eddington's theory is small compared with the variation in density. Consider thus the state of ionised matter in a star which is in isothermal equilibrium. For simplicity the substance will be assumed to consist of free electrons, each with a charge $-e$ and mass m, and free positive ions each with a charge Ze and mass M. Let the gravitational potential be denoted by ϕ and the electrical potential by ψ. We further treat the free electrons and the positive ions as entities of ideal gases. The number of free electrons N_e and of positive ions N_p per unit volume will then, by Boltzmann's principle, be given by expressions of the form

$$N_e = N_e^{\circ} \exp \frac{(-m\phi + e\psi)}{kT} \quad . \quad . \quad . \quad (1)$$

and

$$N_p = N_p^{\circ} \exp \frac{(-M\phi - Ze\psi)}{kT} \quad . \quad . \quad . \quad (2)$$

where T is the absolute temperature of the medium, k Boltzmann's constant, and N_e° and N_p° two constants which will depend on the special structure of the star in question. When ψ can be put equal to zero these equations are nothing but the ordinary expressions for the hydrostatic equilibrium of an isothermal gaseous star under the action of gravity. On the other hand, the electric volume density

$$\rho_e = eZN_p - eN_e \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the mass density

$$\rho = MN_p + mN_e \quad . \quad . \quad . \quad . \quad (4)$$

are obtained from the corresponding equations of Poisson

$$\rho_e = -\frac{1}{4\pi}\nabla^2\psi \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\rho = +\frac{1}{4\pi G}\nabla^2\phi \quad . \quad . \quad . \quad . \quad (6)$$

where G is the gravitational constant.

It will be seen that the above equations are satisfied when the potentials ψ and ϕ stand in the ratio

$$\frac{\psi}{\phi} = -\frac{M-m}{e(Z+1)} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here ϕ is negative, and accordingly ψ positive. The result can be simply interpreted, for granted the state to be electrically neutral initially the free electrons will rapidly spread outwards, owing to their large thermal velocities, and thus induce a positive charge on the star. The net outward diffusion must stop when the inward force on a free electron is equal to the inward force on a positive ion, and this condition is just expressed by (4). There is an interesting consequence involved in this result, for neglecting the mass of an electron compared with the mass of a positive ion the resultant potential energy of an ion, regardless whether it is negative or positive, is found to be

$$E = \frac{M}{Z+1}\phi \text{ *} \quad . \quad . \quad . \quad . \quad (8)$$

This result has so far been derived only for the case of an equilibrium between free electrons and one kind of positive ions. For the case, however, in which the electric charge on a positive ion can be considered to be proportional to the mass, the above result will still be approximately true provided the different values of Z are all large compared with unity. These conditions must be approximately realised in the far interior of the stars, where most of the electrons present will enjoy a free state, while only a minor part of them are really bound to the positive ions. Thus for elements of low atomic number the dissociation into free electrons and free nuclei may be considered to be complete, and in this case Z may be identified with the atomic number of the substance. Now it is an empirical law that the atomic weight of an element is approximately proportional to its atomic number, therefore the potential energy of an atom will by (8) be approximately independent of the atomic number. In consequence of this fact elements of different atomic weights would be expected to be nearly evenly distributed throughout the star even if there are no convective currents present. If gravitational forces alone were operative we should expect a very strong concentration of heavy elements at the centre of the stars, and only the lightest elements in the atmosphere at the surface. The electric forces may therefore be very important for the problem of determining the relative frequency of elements of different atomic weights throughout the star. For the equilibrium of the total mixture of positive ions plus negative electrons, however, they will be negligible, as the additional attraction on the free electrons will very nearly compensate the repulsion on the positive ions. This is very easily seen, for we have from the equations (5), (6), and (7)

$$\rho_e = \frac{GM}{Z+1}\frac{\rho}{e} \quad . \quad . \quad . \quad . \quad (9)$$

* This result was derived in a different manner by A. Pannekoek, *Bull. Astr. Nether* , No. 19, p. 107, 1922.

and the resultant electric force on the matter contained in unit volume f_e is

$$f_e = -\frac{M^2G^2}{e^2(Z+1)^2} f_g \quad . \quad . \quad . \quad . \quad (10)$$

where f_g is the gravitational attraction on the matter contained in unit volume. The ratio f_e/f_g is at once seen to be negligibly small, and further consideration of this case is unnecessary. It must, however, be mentioned that the electric charge as calculated above may be altered appreciably if there is strong emission of α- or β-particles from the surface of the star, but it is not necessarily changed by ordinary evaporisation of atoms and electrons from the surface, since the velocity of evaporisation will be very nearly the same for both kinds of particles when a relation like (7) is fulfilled at the surface of the star. It is difficult to imagine a state of things in which the order of magnitude of the resultant electric charge predicted by (9) should be so seriously departed from that the effect should be of importance for the conditions of the hydrostatic equilibrium, and we neglect this possibility altogether.

3. *The Electrical Pressure.*—We next turn to the examination of the force arising from the polarisation of the medium. The necessary existence of such a force is realised in the following way. Consider a free particle threading its way through the matter of the star. It will constantly exert attractive forces on particles of opposite sign and repel particles of the same sign. The net result will be that in the vicinity of the specified particle there will in the mean be a preponderance of particles of opposite sign and a deficit of particles of the same sign. If density and temperature are constant in the part of space in which the particle moves, the polarisation round the particle must in the mean be radially symmetrical, and there will exist no resultant force on the particle. If, on the other hand, density and temperature vary from point to point, the polarisation can no longer be radially symmetrical. Assuming the temperature to be constant, we shall expect an excess of polarisation on the side towards increasing density, and the medium will be expected to exert a resultant force on the particle in this direction. The force is directed in the same sense for positive as well as for negative particles, and we see at once that the total force will be cohesive and thus contribute a negative term to the expression for the gas pressure in the star.

For an approximate calculation of the total pressure it is in the present case convenient to consider the problem from the point of view of the virial theorem. Consider thus an assembly of atoms and electrons contained in an enclosure of unit volume and at a temperature T. Let $\phi_{ik}(r_{ik})$ be the central force acting between two specified particles i and k, which are at the mutual distance r_{ik}. If p is the average value of the pressure and N the total number of particles contained in the enclosure it can be simply shown that *

$$p = NkT + \tfrac{1}{3}\overline{\sum_{ik} r_{ik}\phi_{ik}(r_{ik})} \quad . \quad . \quad . \quad . \quad (11)$$

The summation in the latter term is to be extended to all combinations of particles, and the bar indicates that the term is to be averaged over

* *Cf.* Jeans, *Dyn. Theory of Gases*, 3rd ed., p. 131.

all possible configurations. For the case of the inverse square law $r_{ik}\phi_{ik}(r_{ik})$ is equal to the relative potential energy of the two particles in question, and the whole term is simply equal to one-third of the mean inner potential energy of the assembly, which we shall denote by U. Throughout the following we shall distinguish between the two terms in (11), and call the first one the gas pressure and denote it by P_G, and the latter term the electrical pressure and denote it by P_E.

The theory of the potential energy of a mixture of ions has been developed by Debye and Hückel * in a recent paper, and we shall take over from this theory as much as is necessary for the purpose. Consider a medium containing a large number of positive and negative ions. We assume the medium to be electrically neutral and the ions to be of two kinds only: (1) free electrons with a charge $-e$, and (2) positive ions each with a charge Ze. Let us fix the attention on a specified positive ion. In the vicinity of this ion, say at a distance r from its centre, there will exist an electric potential the mean value of which we denote by ψ. Let n be the number of positive ions per unit volume; the number of free electrons per unit volume will then be Zn. In a small volume element $d\mathrm{V}$, which is considered to be at the constant distance r from the specified ion the mean density of ions and electrons must be different from the mean values evaluated in a volume element taken at random. A tentative application of Boltzmann's principle to this case leads to the view that the mean number of free electrons in the volume element $d\mathrm{V}$ will be

$$d\mathrm{V}n\mathrm{Z}\exp\left(\frac{e\psi}{k\mathrm{T}}\right) \quad . \quad . \quad . \quad . \quad (12)$$

and the number of positive ions

$$d\mathrm{V}n\exp\left(-\frac{\mathrm{Z}e\psi}{k\mathrm{T}}\right) \quad . \quad . \quad . \quad . \quad (13)$$

By Poisson's equation the potential ψ must satisfy the relation

$$\nabla^2\psi = +4\pi e n\mathrm{Z}\left\{\exp\left(\frac{e\psi}{k\mathrm{T}}\right)-\exp\left(-\frac{\mathrm{Z}e\psi}{k\mathrm{T}}\right)\right\} \quad . \quad . \quad (14)$$

At large distances from the specified ion the field must be nearly zero on an average, and the quantity in the parenthesis can be expanded in a rapidly convergent power series. Retaining only the first term of the expansion the differential equation above assumes the form

$$\nabla^2\psi = \frac{4\pi e^2 n\mathrm{Z}(\mathrm{Z}+1)}{k\mathrm{T}}\psi \quad . \quad . \quad . \quad . \quad (15)$$

Close to the specified ion this approximation undoubtedly breaks down. It is probably very difficult to develop an exact theory, the more so as the above application of Boltzmann's principle may be questioned. It is therefore advisable to develop the theory on the basis of the approximate expression (15), expecting in this manner to obtain the correct order

* *Physikalische Zeitschrift*, **24**, 1, 1923.

of magnitude of the effect. The solution of (15) for spherical symmetry is

$$\psi = \frac{C}{r} e^{-xr} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where

$$x = \left(\frac{4\pi e^2 n Z(Z+1)}{kT} \right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (17)$$

and has the dimension of an inverse length. For positive ions we have to put the constant C equal to Ze, and for electrons equal to $-e$. In (16) is included not only the potential arising from the systematic crowding of the particles in the surrounding medium, but also the potential due to the particle itself. This latter is a pure inverse square field of strength C/r, and by subtraction the potential as it is due to the medium alone is obtained

$$\frac{C}{r}(e^{-xr} - 1) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The ion itself has thus a potential energy relative to the gaseous substance equal to

$$-C^2 x \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the total potential energy of the assembly is

$$U = -\tfrac{1}{2}\Sigma C^2 x \quad . \quad . \quad . \quad . \quad (20)$$

where the summation is to be extended to all particles of the assembly. The electrical pressure is

$$P_E = -\tfrac{1}{3}\left[\frac{\pi e^6 Z^3 (Z+1)^3 \rho^3}{k H^3 A^3 T} \right]^{\frac{1}{2}} \quad . \quad . \quad . \quad (21)$$

where we have introduced

$$n = \frac{\rho}{AH}$$

in which A is the true atomic weight of the substance and H the mass of a hydrogen atom. For very dense substances it would have been necessary to take account of the fact that at very close collisions the inverse square law is no longer applicable,* provided the atoms do not consist of bare nuclei, but also contain bound electrons. It is not accurately known at what mutual distance between two atomic ions the

* Cf. Debye and Hückel, *Phys. Zeitschr.*, 24, 8, 1923. A rigorous treatment would also have to take account of the fact that in the close vicinity of an ion the motion of an electron will be restricted by quantum conditions provided its energy is sufficiently small. This effect can for the present case only be expected to be of importance for electrons moving in orbits with small quantum numbers, as the higher quantum states must be so strongly broadened by action of contiguous particles that the co-ordinates and velocities of the electrons may be considered to be continuously distributed in phase. For the above case the whole question is largely irrelevant, as the main contribution to the potential Cx comes from particles which are situated beyond a distance x^{-1} from the particle, and for the typical stellar case the value of x^{-1} is so large compared to the probable dimensions of the positive ions that the application of classical statistics seems secure.

inverse square law ceases to be applicable, but it is natural to expect the effect to be perceptible as soon as the distance is small enough for a polarisation of the electric system to be perceptible. Some information regarding this question can be obtained from experiments with neutral helium, as the dimensions of the electronic system of the helium atom may be regarded as known from atomic theory, the mean radius of the electronic system being probably close to $3 \cdot 10^{-9}$ cm. On the other hand, a value for the distance between two helium atoms at which the interatomic fields of force become perceptible, can be derived from experiments on the van der Waals correction in volume, giving for the mean radius of the helium atom the value $9 \cdot 9 \cdot 10^{-9}$ cm. This value is seen to be three times larger than the value for the true dimensions of the electronic orbits. By analogy it will be expected that the radius of any atomic ion in the above sense will be comparable with the dimensions of the electronic orbits of the ions, although somewhat larger.

4. In the following the possible correction due to the bound electrons will be neglected, the effect of the forces due to the residual charges on the ions alone being considered. Denoting the pressure of radiation by P_R, which by Stefan's law is given by

$$P_R = \frac{a}{3} T^4$$

where a is Stefan's constant, the equation of hydrostatic equilibrium assumes the form

$$\frac{d}{dr}(P_G + P_R + P_E) = -\rho g \quad . \quad . \quad . \quad (22)$$

where g is the gravitational acceleration.

Before proceeding further it is convenient to review shortly Eddington's theory of the hydrostatic equilibrium of a star. The electrical pressure is then to be neglected and the gravitational force ρg is assumed to be proportional to the gradient of light pressure, the constant of proportionality being denoted by $(1-\beta)^{-1}$. Neglecting the very small residual light pressure at the surface of the star, this assumption entails the relation

$$P_G = \frac{\beta}{1-\beta} P_R \quad . \quad . \quad . \quad . \quad (23)$$

From this expression it follows directly that the density is proportional to the temperature in the third power. The total pressure is

$$P = P_G + P_R = \frac{1}{1-\beta} P_R$$

thus being proportional to the temperature in the fourth power, and the relation between pressure and density is

$$P = \left\{ \frac{3k^4(1-\beta)}{a\mu^4 H^4 \beta^4} \right\}^{\frac{1}{3}} \rho^{4/3} = \tau \rho^{4/3} \quad . \quad . \quad . \quad (24)$$

where μ is the mean molecular weight entering the expression of the gas pressure.

By simple dimensional reasoning it follows that, thanks to the special value of the exponent in (24), the quantity τ will be independent of the scale on which the star is constructed, and only depend on the total mass M of the star in the combination

$$\tau = \text{const}\, M^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The quantity β is therefore determined by the quartic equation

$$1-\beta = \text{const}\, \mu^4 M^2 \quad . \quad . \quad . \quad . \quad (26)$$

where the value of the numerical multiplier is 0·00308 when the sun's mass is taken as unity.*

We now turn to the generalisation of this theory for the case that the electric pressure is not to be neglected. Assuming, as before, that the gravitational force is proportional to the gradient of radiation pressure, equation (22) transforms into

$$\frac{d}{dr}\left(P_G - \frac{\beta}{1-\beta}P_R + P_E\right) = 0$$

or

$$P_G - \frac{\beta}{1-\beta}P_R + P_E = 0 \quad . \quad . \quad . \quad . \quad (27)$$

which is the general relation between ρ and T. Introducing the proper values for the different terms in this expression, it will be found that also for this case the density is proportional to the temperature in the third power and the total pressure proportional to the temperature in the fourth power. Consequently the relation between pressure and density will have the same form as in the case considered above, and by a short calculation it is found to be

$$P = \left\{\frac{3k^4}{a\mu^4 H^4}\frac{1-\beta}{[\beta + x(1-\beta)]^4}\right\}^{\frac{1}{3}} \rho^4/_3 \quad . \quad . \quad . \quad (28)$$

where x, which is the ratio of electrical pressure to radiation pressure, is given by the relation

$$x^2 q_1 - \left(x + \frac{\beta}{1-\beta}\right)^3 = 0 \; ; \quad q_1 = \frac{27k^4}{\pi a e^6 Z^3} \quad . \quad . \quad (29)$$

where we have made use of the approximate relation

$$\mu = \frac{A}{Z+1} .$$

Thus Eddington's fundamental equation (26) is for the present case to be replaced by

$$1-\beta = 0{\cdot}00308\mu^4 M^2[\beta + x(1-\beta)]^4 \quad . \quad . \quad . \quad (30)$$

or eliminating x

$$1-\beta = q_2(\beta + [q_2^3 q_1^4(1-\beta)]^{-\frac{1}{3}})^4 \; ; \quad q_2 = 3{\cdot}0810^{-3}\mu^4 M^2 \quad . \quad (31)$$

* A. S. Eddington, *M.N.*, 84, 109.

While β on Eddington's theory is a function of the mean molecular weight and total mass of the star, it will on the present extension of the theory also depend on the number of electric charges carried by each positive ion in the star. Eddington's conclusion that the luminosity of a star of constant mass will remain nearly constant during its evolution would therefore, in the most general case, only be true provided the degree of ionisation could be considered to remain constant. This fact, however, is scarcely of much importance, as it will appear that the absolute value of the electrical pressure in all practical cases will be so small compared with that of gas or even light pressure as to be negligible altogether. Consider thus as a convenient measure of the correction to be introduced the value of Z for which the electrical pressure will just counterbalance light pressure ($x = 1$), the state then being as if the star were built up of an ideal gas in purely gravitational equilibrium. Solving (29) for Z we have on using (31)

$$Z = x^{\frac{2}{3}} q_2^{\frac{1}{4}} (1-\beta)^{\frac{3}{4}} \left(\frac{27k^4}{\pi a e^6}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad (32)$$

Consider, for example, a star of mass $1{\cdot}5\odot$ and a mean molecular weight 2·8. The proper value of β for this star for $x = 1$ is $\beta = 0{\cdot}836$. Introducing numerical values for the constants in (32) we obtain

$$Z = 65.$$

When we realise that atoms with a residual charge $Z = 65$ would be likely to retain an appreciable number of bound electrons, it will appear that the case considered above could only occur if the star were built up of the heaviest elements known from terrestrial observations, and for elements of median or small atomic numbers the electric pressure would be expected to be quite unimportant.

It must be emphasised that the above calculation can only be expected to have approximate validity because of the somewhat doubtful application of Boltzmann's principle underlying the theory, and because of the approximation leading to equation (15) which lies at the base of the theory. Especially the latter circumstance indicates that the above results must be regarded as a lower limit to the true effect. It is, however, doubtful whether the result would be improved by taking into account higher terms on the right-hand side of (15), and a rigorous theory would have to apply Boltzmann's principle *ab initio* to a general assembly of particles in the manner attempted by Milner.*

Institute of Theoretical Physics,
Copenhagen: June 20.

* *Phil. Mag.*, 23, 551, 1912.

Note on the Employment of Angular Variables in Celestial Mechanics. By T. M. Cherry, B.A., Isaac Newton Student of the University of Cambridge.

(*Communicated by Professor H. F. Baker, F.R.S.*)

In the methods of Lindstedt and Delaunay for expressing the general solution of the Problem of Three Bodies in purely trigonometric form a typical feature is the employment of canonic co-ordinates of which half are angular variables. The object of this note is to show that, if the Cartesian co-ordinates of the bodies are expressible in terms of t by uniformly convergent multiple Fourier series, the angular co-ordinates are in general given by more complicated series containing "small divisors"; so that by employing a method of solution using angular co-ordinates, the problem would be given an artificial complication. The conclusion, therefore, is that *the use of angular variables in Celestial Mechanics is fundamentally unsuitable.*

Let $\xi_1, \ldots \xi_n, \eta_1, \ldots \eta_n$ be the co-ordinates in a mechanical problem of which the solution is expressible in the form

$$\left.\begin{aligned}\xi_\nu &= \phi_\nu(m_1 t + \epsilon_1, m_2 t + \epsilon_2, \ldots m_n t + \epsilon_n, a_1, \ldots a_n)\\ \eta_\nu &= \psi_\nu(m_1 t + \epsilon_1, m_2 t + \epsilon_2, \ldots m_n t + \epsilon_n, a_1, \ldots a_n)\end{aligned}\right\} \quad . \quad (1)$$

where ϕ_ν, ψ_ν are analytic functions, periodic in each of their first n arguments with period 2π, the ϵ's and a's are constants of integration, and the m's are analytic functions of the a's. Let $x_1, \ldots x_n, y_1, \ldots y_n$ be a new set of co-ordinates, of which $y_1, \ldots y_n$ are angular variables; to make the problem definite we shall suppose these connected with the old co-ordinates by the relations

$$\left.\begin{aligned}\xi_\nu &= \sqrt{2x_\nu}\cos y_\nu\\ \eta_\nu &= \sqrt{2x_\nu}\sin y_\nu\end{aligned}\right\} (\nu = 1, \ldots n) \quad . \quad . \quad . \quad (2)$$

which transform a conjugate pair of variables ξ_ν, η_ν into a conjugate pair x_ν, y_ν. We shall consider the expression of y_ν as a function of t given by (1), (2).

From (2) we have

$$\tan y_\nu = \frac{\eta_\nu}{\xi_\nu},$$

and thus

$$\frac{dy_\nu}{dt} = \frac{\xi_\nu \dfrac{d\eta_\nu}{dt} - \eta_\nu \dfrac{d\xi_\nu}{dt}}{\xi_\nu^2 + \eta_\nu^2};$$

hence from (1) $\dfrac{dy_\nu}{dt}$ is expansible in a convergent series of the form

$$\frac{dy_\nu}{dt} = A_0 + \Sigma A_{p_1, \ldots p_n} \cdot \exp\left(p_1 m_1 + p_2 m_2 + \ldots + p_n m_n\right)t \quad . \quad (3)$$

where the A's are analytic functions of $a_1, \ldots a_n$ and the summation is over all integral (positive, negative, and zero) values of $p_1, \ldots p_n$. This gives on integration

$$y_\nu = A_0 t + \Sigma \frac{A_{p_1, \ldots p_n} \cdot \exp(p_1 m_1 + p_2 m_2 + \ldots + p_n m_n) t}{p_1 m_1 + p_2 m_2 + \ldots p_n m_n} \quad . \quad (4)$$

and A_0 may thus be called the mean motion of the co-ordinate y_ν. The expression (4) is, however, only valid when the constants of integration $a_1, \ldots a_n$ have such values that $m_1, \ldots m_n$ are connected by no homogeneous relation of the form

$$p_1 m_1 + p_2 m_2 + \ldots p_n m_n = 0 \quad . \quad . \quad . \quad (5)$$

in which $p_1, \ldots p_n$ are integers not all zero, for otherwise a denominator in (4) will vanish. Should $a_1, \ldots a_n$ have such values that relations of the form (5) exist, we must before integration replace (3) by an equation of the form

$$\frac{dy_\nu}{dt} = A_0' + \Sigma A'_{p_1, \ldots p_{n'}} \cdot \exp(p_1 m_1' + p_2 m_2' + \ldots + p_{n'} m'_{n'}) t \, . \quad (6)$$

in which $n' < n$; $m_1', \ldots m'_{n'}$ are connected by no relation of the form (5); and in general $A_0' \neq A_0$, since A_0' consists of the sum of A_0 and the coefficients of those terms from the right of (3) whose arguments have become independent of t in virtue of the relations connecting $m_1, \ldots m_n$.* From (6) we see that the mean motion of y_ν is now A_0'.

Thus the mean motion of the angular co-ordinate y_ν *is not in general a continuous function of the constants of integration* $a_1, \ldots a_n$, *but is discontinuous at all such sets of values as make* $m_1, \ldots m_n$ *connected by a homogeneous linear relation with integral coefficients.* In general such sets are everywhere dense in the domain of (real) values of $a_1, \ldots a_n$. From (1) the a's are continuous functions of the initial values of the ξ's and η's, and hence from (2) of the initial values of the x's and y's; thus the mean motion of y_ν is a discontinuous function of the initial values of the x's and y's.

The formal solution of the Problem of Three Bodies is usually presented † in the form

$$\left.\begin{aligned} x_\nu &= f_\nu(w_1, \ldots w_n) \\ y_\nu &= w_\nu + g_\nu(w_1, \ldots w_n) \end{aligned}\right\} (\nu = 1, \ldots m)$$

in which

$$w_\nu = m_\nu t + \epsilon_\nu$$

and f_ν, g_ν are multiple Fourier series with the arguments $w_1, \ldots w_n$. In the light of what has just been seen, it is evident that such series cannot be uniformly convergent over any range, however small, of the

* In writing (1) with n independent arguments we imply that $m_1, \ldots m_n$, which are functions of $a_1, \ldots a_n$, are connected by no *identical* relation of the form (5).

† *E.g.* in the methods of Delaunay (Tisserand, *Mécanique Céleste*, t. iv., chap. xxvi.), Lindstedt (Poincaré, *Méthodes Nouvelles de la Mécanique Céleste*, chap. ix.), Whittaker (*Analytical Dynamics*, chap. xvi.).

initial values of the co-ordinates, on which the mean motions $m_1, \ldots m_n$ depend; for $m_1, \ldots m_n$ are discontinuous functions of these initial values, and since $w_1, \ldots w_n$ are used as the arguments in the series f_ν, g_ν, the coefficients in these series must also be discontinuous functions. Indeed, in Lindstedt's series, as in the series (4) above, there occur divisors which depend on the constants of integration, and since some of the divisors vanish in any range of these constants, the series are not even formally valid over any finite range of these constants: it may plausibly be suggested that this peculiar complication arises from the use of angular variables.

The equations of Celestial Mechanics initially present themselves in a form in which none of the co-ordinates are angular, and the object of so transforming them that half the co-ordinates become angular is presumably to throw them into a form as nearly as possible allied to the desired trigonometric form of the solution. The above note has shown that in general this procedure must defeat its own object.

The "Arcus Visionis" of the Planets in the Babylonian Observations.

By C. Schoch.

(*Communicated by J. K. Fotheringham, D.Litt.*)

In his recently published book* Father Kugler has stated that the Babylonian astronomers of the last four centuries before Christ always put the heliacal setting of Sirius on May 14 Julian and the heliacal rising on July 19, if we reckon the day from midnight. The *arcus visionis* (in this paper called γ) of a planet or of Sirius is the depression of the Sun below the horizon, measured in the vertical circle for the moment when the star sets on the last evening when it is visible or rises on the first morning when it is visible, refraction being disregarded in the case of both bodies. In the case of the outer planets and fixed stars the heliacal setting will be called elast (evening last) and the heliacal rising mfirst (morning first). Father Kugler has computed that the mfirst of Sirius at Babylon (latitude $= 32^\circ{\cdot}5$) occurred on July 19 with $\gamma = 7^\circ{\cdot}4$. I have computed several mfirsts of Sirius and have found for γ a mean value of $7^\circ{\cdot}8$. This value for mfirst is in striking contradiction to the value used by Ptolemy ($\gamma = 11^\circ$), and adopted by all astronomers and chronologists (Ideler, *Handbuch der Chronologie*; Ginzel, *Handbuch der Chronologie*; Neugebauer, *Tafeln zur astron. Chronologie*). This difference of $3^\circ{\cdot}2$ in γ is increased if we compute γ for elast of Sirius. I find a mean value of $6^\circ{\cdot}5$, which is smaller than Ptolemy's value (11°) by $4^\circ{\cdot}5$. The Babylonians could therefore observe the elasts of Sirius five days later than Ptolemy, the mfirsts four days earlier. The cause of this enormous difference can be found only in the circumstance that at Alexandria there

* *Sternkunde und Sterndienst in Babel*, II. Buch, II. Teil, 2 Heft, 1924, p. 523.

was always a layer of mist covering the lower part of the horizon, even when the rest of the sky was quite clear. On the other hand, at Babylon (and Athens) the lower part of the horizon was free and the brighter stars were visible till rising or setting. It is quite impossible to assume that the Babylonians were in error, because their observations of Mercury and Jupiter are in full agreement with those of Sirius.

I use the following abbreviations for the phenomena of the inner planets, Mercury and Venus. After superior conjunction the planet becomes visible in the evening (efirst), disappears in the evening before inferior conjunction (elast), then reappears after inferior conjunction in the morning (mfirst), and disappears in the morning before superior conjunction (mlast). Now, for Mercury so many observations have been published by Father Kugler (*Sternkunde und Sterndienst in Babel*, I. Buch, 1907) and still more by Epping (*Zeitschrift für Assyriologie*, vols. v.–viii., 1890–93), that I could compute the angle γ for 30 Babylonian dates. The result is that for all four phenomena of Mercury the angle γ was different, agreeing well with the stellar magnitude and the conditions of observation ("lasts" or "firsts"). While in our latitudes it is not easy to see Mercury with the naked eye, the Babylonians as far back as the year -567 noted on Airu 10 an efirst, and the value γ of this observation corresponds accurately with that of the later observations of efirst.

All my computations are based on observations, not on Babylonian prediction-tablets. I have used: for Mercury 30 observations, for Venus 15, for Jupiter 12, for Saturn 9, for Mars only 4. For this planet therefore the result is uncertain. As the computations to obtain the angle γ for about 70 observations are very intricate, I have used my MS. tables for geocentric places of the planets, constructed several years ago. In these tables the following sidereal accelerations are assumed: Earth $+0''\cdot89s^2$, Mercury $+4''\cdot11s^2$, Venus $+1''\cdot65s^2$, Mars $+0''\cdot45s^2$, where s signifies the number of centuries elapsed since 1900. The total corrections applied to Newcomb's mean longitudes are:

$$\begin{aligned}
&\text{Earth} \quad +0''\cdot51+1''\cdot08s+0''\cdot907s^2\\
&\text{Mercury}+1''\cdot62+3''\cdot60s+4''\cdot133s^2\\
&\text{Venus} \quad +0''\cdot61+1''\cdot12s+1''\cdot645s^2.
\end{aligned}$$

The resultant corrections to Newcomb's mean heliocentric longitudes for 1925 are: Earth $+1''$, Mercury $+3''$, Venus $+1''$. The effect of these accelerations is to place ancient conjunctions of Mercury and Venus with the Sun earlier than the times that Newcomb's tables would give. The value of γ is dependent: (1) on the right ascensions and declinations of the Sun and the star; (2) on the stellar magnitude (m) of the star; (3) on the difference in azimuth (A) of the Sun and star, all computed for the moment of efirst, mlast, etc.* The greater the magnitude is, the smaller γ is, and the greater A is, the smaller γ is. A is always small for the planets, but in the case of Sirius it is very great, 51°. So γ for Sirius must be smaller than for a planet with the same value of m. But the

* See Vogt, *Sitzungsberichte der Heidelberger Akademie, philosoph.-hist. Classe*, 1920, Abhandl. 15, p. 8.

Babylonian observations prove also that all "lasts" had a smaller γ than the "firsts." The Babylonians were persistent observers, and it was comparatively easy for them to find the place of a star, known by the observation of the previous day (the case of the "lasts"). For a reappearance (the case of the "firsts") the place of a star was unknown and not given by previous observations.

Mercury.—m is computed approximately. All 30 observations show that γ was smaller for superior conjunction (mlast, efirst) than for inferior conjunction (elast, mfirst), in full agreement with the value of m. All "lasts" have a smaller γ than the "firsts". The minimum of mlast is $\gamma = 9°{\cdot}0$.

Venus.—m is equal for superior and inferior conjunction, but the observations seem to give a slightly smaller γ for elast and mfirst than for mlast and efirst. The mean value is $5°{\cdot}2$ to $5°{\cdot}8$; the minimum $4°{\cdot}9$ elast, the sky probably being very clear during this observation. The value given by Dr. Neugebauer,* 7°, is far too high. Of the greatest importance is the interval of invisibility at inferior conjunction, between elast and mfirst. It varies between 2 and 19 days for Babylon and depends only on the geocentric latitude of Venus. The smaller intervals of 2 or 3 days can recur on the same Babylonian date only after periods of 56, 64, or 120 years, and they provide a very good means of identifying the Babylonian year where such observations are recorded. The first observations of Venus were probably made as far back as 3000 years before Christ.

Jupiter.—For elast γ is much smaller than for mfirst. Jupiter has at conjunction the same m as Sirius, but it is easier to observe Sirius, because the difference of azimuth Sun – Sirius is 51°; hence γ is smaller for Sirius so as to affect the time of mfirst and elast by nearly 2 days. Dr. Neugebauer's value, 7°, is too small. It is strange that he attributes to Venus and Jupiter the same value of 7° although their brilliance differs by nearly two stellar magnitudes.

Sirius has been already discussed. Elast in the time from −300 to +100 for Babylon fell on May 14 (with Ptolemy's false value of γ on May 10). Mfirst fell on July 19 (with Ptolemy's value on July 23), for Memphis (latitude = 30°) with the value of γ as determined for Babylon on July 17 (with Ptolemy's value on July 21).

Saturn.—The value of m depends on whether the planet's ring is open or not, and may vary as much as half a magnitude from the mean value which I tabulate. γ for elast is much smaller than for mfirst.

Mars.—m is right, for the stellar magnitude at conjunction is subject to little variation; at opposition, however, m varies between $-2{\cdot}7$ and $-0{\cdot}9$. (At the opposition of this year, 1924 August, m is $-2{\cdot}6$; that is the most favourable opposition within the next hundred years.) To determine γ I had only 4 observations, which are quite insufficient.

The following table gives the approximate stellar magnitude m and the *arcus visionis* γ, arranged according to the value of m. All values of m and γ are mean values; m is computed for the time of elast, etc., not for the time of conjunction.†

* *Tafeln zur astron. Chronologie*, Heft 3, 1922, p. 60.

† See the formula given in Müller, *Photometrie der Gestirne*.

Babylon. Observ.	m	γ
Venus, inf. conj.:		°
elast	−3·2	5·2
mfirst	−3·2	5·7
sup. conj.:		
mlast	−3·3	5·8
efirst	−3·3	5·8
Jupiter:		
elast	−1·6	7·4
mfirst	−1·6	9·3
Sirius:		
elast	−1·5	6·5
mfirst	−1·5	7·8

Babylon. Observ.	m	γ
Mercury, sup. conj.:		°
mlast	−0·5	9·5
efirst	−0·5	10·5
inf. conj.:		
elast	+1·2	11·1
mfirst	+1·2	13·2
Saturn:		
elast	+0·9	10·5
mfirst	+0·9	13·0
Mars:		
elast	+1·8	14·2
mfirst	+1·8	15·5

For places with a very clear sky (such as Babylon and Athens) I recommend the use of the following values of γ for stars near the ecliptic (A not greater than 25°):—

m.	γ	
	Lasts.	Firsts.
	°	°
−3·0	5·8	6·5
−2·0	7·0	8·0
−1·0	8·5	9·5
0·0	9·5	10·5
+1·0	10·5	11·5
+2·0	12·5	13·5
+3·0	15·0	16·0

From the constants given above I have computed the following dates for the phases of Sirius and summer solstice at Babylon, time being reckoned from Babylon midnight:—

Year.	Elast.	Mfirst.	Solstice.
0	May 14	July 19	June 24·8
−1000	,, 11	,, 19	July 2·9
−2000	,, 8	,, 18	,, 10·7
−3000	,, 5	,, 18	,, 18·3

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1924 *July* 11.

The Stellar Magnitude Scales of the Astrographic Catalogue. Eighteenth Note: The San Fernando Magnitudes for −4° and −6°, and the Algiers Magnitudes for −1°. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. Sr. Herrero has kindly furnished counts for zones −4° and −6°, following on those previously sent for −3° and −5° and published in the seventeenth note (*M.N.*, **83**, 385). These are given in Tables I. and II. below. Cumulative counts were formed from them up to and

TABLE I.

Counts of Zone −4° (San Fernando) for each hour. Numbers of plates, 8, 7, 8, 7, etc.

h	]8·5.	8·5.	9·0.	9·5.	10·0.	10·5.	11·0	11·5.	[11·5.	All.
0	48	34	74	142	182	307	228	134	8	1157
1	35	37	54	108	199	241	205	91	6	976
2	43	33	85	139	210	307	253	166	6	1242
3	47	39	72	119	201	283	226	193	2	1182
4	65	82	112	181	306	456	310	188	36	1736
5	63	62	114	185	250	366	333	429	201	2003
6	81	119	224	352	671	1020	936	672	100	4175
7	73	113	253	505	1036	1363	1000	489	54	4886
8	73	104	217	416	582	690	478	220	19	2799
9	38	91	138	248	336	243	120	48	2	1264
10	36	54	121	229	318	264	165	74	5	1266
11	33	43	84	192	263	262	148	56	8	1089
12	41	45	106	203	319	326	146	78	10	1274
13	32	43	102	216	339	404	279	153	26	1594
14	31	57	95	204	259	307	192	57	12	1214
15	33	35	98	179	252	219	122	38	6	982
16	47	54	118	280	340	288	144	34	29	1334
17	55	28	86	176	294	245	142	74	11	1111
18	64	70	134	227	331	299	145	94	14	1378
19	59	69	131	278	376	372	198	142	9	1634
20	55	80	149	412	770	799	579	237	26	3107
21	45	50	95	222	406	385	228	119	10	1560
22	41	51	106	197	320	266	210	70	25	1286
23	32	40	64	129	260	312	212	123	20	1192
Sum	1170	1433	2832	5539	8820	10024	6999	3979	645	41441

TABLE II.

Counts of Zone −6° (San Fernando) for each hour.
Numbers of plates, 8, 7, 8, 7, etc.

h	]8·5.	8·5.	9·0.	9·5.	10·0.	10·5.	11·0.	11·5.	[11·5.	All.
0	46	47	63	107	170	194	137	95	31	890
1	36	36	59	95	173	174	160	89	49	871
2	47	38	90	168	257	282	224	169	48	1323
3	45	47	93	169	221	186	138	91	19	1009
4	62	68	117	147	251	307	309	335	141	1737
5	79	81	119	163	254	299	237	184	105	1521
6	107	95	242	346	491	672	619	720	191	3483
7	75	149	242	369	550	786	804	1061	375	4411
8	94	99	172	277	374	494	468	504	391	2873
9	44	48	85	156	224	222	221	122	29	1151
10	38	49	84	164	193	219	193	169	37	1146
11	39	48	63	94	111	124	152	124	40	795
12	50	38	96	137	186	217	188	184	45	1141
13	48	39	62	123	178	184	203	122	28	987
14	64	45	97	195	247	211	235	145	42	1281
15	57	45	111	190	275	274	183	131	48	1314
16	40	48	102	203	251	327	278	210	54	1513
17	46	33	63	116	149	156	236	173	46	1018
18	78	87	116	190	305	481	611	674	199	2741
19	65	64	100	184	344	322	401	407	115	2002
20	78	47	81	161	281	351	506	671	318	2494
21	46	43	75	139	203	265	293	319	129	1512
22	50	55	86	110	207	269	233	142	32	1184
23	39	35	58	101	195	181	221	186	53	1069
Sum	1373	1384	2476	4104	6090	7197	7250	7027	2565	39466

including 8·5, 9·5, 10·5, and All (as for −3° and −5°). Of these the logarithms were taken, and after subtracting log 4 (the number of square degrees per plate), and log 8 or log 7 (number of plates in an hourly group) are considered to be values of log N_γ at Galactic Latitude γ, from which log N_0 is deduced by the formula

$$\log N_0 = (1 + \cdot 0025\gamma) \log N_\gamma + K_\gamma$$

(see *M.N.*, 78, 54). The resulting mean values of log N_0 are shown in Table IV.; in Table III. the differences from these are given, providing the material for analysis in Table IV.

TABLE III.

San Fernando Residuals after Correction for Galactic Latitude.

(The unit is 0·01 in log N_0.)

	Zone −4°.					Zone −6°.				
h	Gal. Lat. °	8·5	9·5	10·5.	All.	Gal. Lat. °	8·5.	9·5.	10·5.	All.
0	−65·5	+ 4	+ 2	+ 6	+11	−67·0	+ 9	+ 3	+ 7	0
1	−63·0	+ 4	− 4	+ 4	+ 7	−65·0	+ 3	+ 2	+ 8	+ 5
2	−55·0	− 3	− 2	+ 1	+ 7	−56·5	+ 2	+14	+19	+15
3	−43·5	+ 5	− 5	− 2	+ 4	−44·5	+ 7	+14	+10	0
4	−30·5	+12	− 4	− 3	− 1	−31·5	+ 6	+ 1	+ 1	+ 5
5	−17·5	− 6	−18	−23	− 9	−18·5	+ 3	− 6	−12	−17
6	− 4·5	− 6	−13	− 9	− 1	− 5·5	− 6	− 4	− 6	− 4
7	+ 8·5	0	+ 5	+14	+17	+ 7·5	+ 6	+ 6	+ 8	+14
8	+21·5	+ 9	+12	+ 9	+ 7	+20·5	+10	+ 7	+ 5	+10
9	+34·0	+17	+15	+ 5	− 4	+33·0	− 3	0	− 1	− 7
10	+45·0	+ 1	+11	+ 6	+ 1	+44·0	− 2	+ 5	+ 2	− 1
11	+53·5	+ 4	+13	+11	+ 6	+52·0	+ 7	0	− 8	− 7
12	+57·5	+ 4	+14	+16	+10	+55·5	+ 3	+10	+ 9	+ 8
13	+55·5	+ 5	+19	+25	+27	+54·0	+ 9	+ 6	+ 7	− 5
14	+48·5	+ 2	+ 8	+ 6	+ 2	+47·0	+10	+15	+11	+ 7
15	+38·5	− 9	+ 1	− 4	−10	+37·5	+ 7	+15	+14	+ 7
16	+26·5	−11	− 3	−12	−19	+25·5	−20	− 9	− 8	−11
17	+13·5	−30	−33	−37	−42	+12·5	−36	−41	−47	−44
18	+ 0·5	−24	−35	−44	−50	− 0·5	−27	−30	−27	−17
19	−12·5	−12	−15	−22	−27	−13·5	− 2	−18	−17	−13
20	−25·5	+ 8	+11	+19	+19	−26·5	− 2	−11	− 5	+13
21	−38·5	+ 7	+ 9	+15	+12	−39·5	+ 3	+ 3	+ 7	+15
22	−50·5	+ 5	+10	+ 8	+ 6	−52·0	+10	+ 4	+ 9	+ 5
23	−60·5	+ 4	+ 2	+13	+17	−62·0	+ 4	+ 3	+10	+14

TABLE IV.

Harmonic Analysis for −4°.

log N_0.	sin θ.	cos θ.	sin 2θ.	cos 2θ.	sin 3θ.	cos 3θ.	sin 4θ.	cos 4θ.
0·87	+ ·071	+ ·002	− ·022	+ ·083	− ·035	− ·020	− ·010	− ·058
1·54	+ ·030	− ·060	− ·035	+ ·135	− ·053	+ ·004	·000	− ·085
2·01	+ ·052	− ·023	− ·028	+ ·162	− ·072	+ ·006	+ ·031	− ·085
2·15	+ ·099	+ ·014	− ·010	+ ·150	− ·098	+ ·010	+ ·050	− ·077
1·75	+ ·050	+ ·049	+ ·019	+ ·094	− ·082	+ ·032	+ ·091	− ·019
1·00	+ ·029	+ ·028	+ ·011	+ ·054	− ·047	+ ·018	+ ·052	− ·011

TABLE IV.—*continued.*

Harmonic Analysis for −6°.

log N_0.	sin θ.	cos θ.	sin 2θ.	cos 2θ.	sin 3θ.	cos 3θ.	sin 4θ.	cos 4θ.
0·89	+·070	+·008	+·003	+·083	−·077	−·003	+·020	−·054
1·47	+·086	−·020	+·045	+·099	−·057	+·024	−·008	−·078
1·88	+·073	+·018	+·038	+·114	−·053	+·038	·000	−·091
2·12	+·053	+·038	−·009	+·085	−·042	+·023	+·025	−·080
1·64	−·030	+·068	−·019	+·017	+·039	+·040	+·013	−·039
1·00	−·018	+·042	−·012	+·010	+·024	+·024	+·008	−·024

At the foot of each part of Table IV. the sum of the first two lines is subtracted from the sum of the second two, showing the changes for 1·75 and 1·64 magnitudes. The change per magnitude is found, in the lowest lines, by simple division by these numbers.

Algiers −1°.

The counts for the adjacent zone −1° photographed at the Algiers Observatory have already been published in *M.N.*, **72**, 700, though unfortunately under the heading +1°, which should have been −1°. At the date of publication (1912 June) the present procedure had not been adopted, but it has since been applied to these counts, and the resulting figures are given below in Tables V. and VI. To obtain the rate of change per unit of log N_0 in Table VI., the second and third lines are subtracted from the sixth and fifth, giving an interval of 2·24 in log N_0. Dividing by this we get the change per unit of log N_0, as in the last line.

In Table VII. these rates of change are collected for the zones −1°, −3°, −4°, −5°, and −6°, and there seems to be no good reason against taking the mean of the five, which are fairly consistent.

We have directed attention chiefly to the first harmonic, the mean value of which is

$$-\cdot002 \sin\theta + \cdot038 \cos\theta = -\cdot038 \cos(\theta - 177^\circ).$$

This differs very widely from the corresponding expression for the nearest zones south of those here treated, viz. −15° to −19° (as in *M.N.*, **81**, 531), which gives

$$-\cdot016 \sin\theta - \cdot077 \cos\theta = -\cdot079 \cos(\theta - 12^\circ),$$

and the relation between the two can only be studied profitably when counts for intermediate zones become available. When the phases at present determined are plotted on a diagram it seems difficult to draw a single smooth curve through them: they suggest, so to speak, not one spiral as formerly proposed, but two. There is nothing unreasonable in the existence of two spiral arms, and when both affect the same zones we might expect the disturbance to be less simple than can be expressed by the first harmonic. In this connection the rise in importance of the

TABLE V.

Algiers Residuals for −1° after Correction for Galactic Latitude.

(The unit is 0·01 in log N_0.)

h	Gal. Lat. °	7·0.	8·0.	9·0.	10·0.	11·0.	12·0.
0	−62·5	− 7	−28	− 9	−16	− 3	0
1	−60·5	−17	−48	−16	−17	− 9	− 9
2	−53	−11	−43	−19	−13	− 9	− 1
3	−42	−13	−13	−11	−10	− 1	− 1
4	−29·5	− 1	− 2	− 3	− 1	− 9	+ 6
5	−16·5	+29	+18	+ 2	− 2	− 9	−11
6	− 3	+21	+15	+19	+27	+10	+22
7	+10·5	+ 9	+ 9	+29	+26	+24	+34
8	+23·5	+14	+ 8	+19	+23	+24	+23
9	+36	−14	+ 5	− 8	− 7	− 7	− 3
10	+47·5	+ 9	+25	+11	+ 7	+ 7	+ 1
11	+56·5	+23	+38	+23	+22	+23	+16
12	+60·5	+16	+17	+11	+10	+ 9	+ 3
13	+58·5	+ 2	+ 3	+11	+10	+17	+ 5
14	+51	0	0	− 3	+ 2	+ 8	+ 6
15	+40	+10	+ 3	− 6	− 4	− 5	− 6
16	+28	−14	0	− 1	− 2	− 3	− 7
17	+15·5	−26	−28	−31	−32	−35	−37
18	+ 2	−30	−28	−36	−41	−49	−58
19	−11·5	−16	− 4	−18	−16	−18	−14
20	−24·5	+ 9	+ 6	+17	+21	+22	+23
21	−37	+ 5	+24	+21	+23	+21	+17
22	−48·5	+19	+18	− 1	+ 1	− 2	− 3
23	−58	−18	+ 1	− 5	− 3	−11	−10
Const.		05	49	126	150	189	210

TABLE VI.

Harmonic Analysis for − 1° (Algiers).

log N_0.	sin θ.	cos θ.	sin 2θ.	cos 2θ.	sin 3θ.	cos 3θ.	sin 4θ.	cos 4θ.
0·05	+ ·078	− ·071	− ·058	+ ·014	− ·138	− ·020	− ·008	+ ·011
0·49	+ ·028	− ·142	− ·138	+ ·032	− ·162	− ·041	− ·093	+ ·008
1·26	+ ·078	− ·082	− ·085	+ ·042	− ·123	+ ·007	+ ·013	− ·032
1·50	+ ·082	− ·077	− ·082	+ ·038	− ·137	− ·002	+ ·021	− ·048
1·89	+ ·075	− ·075	− ·059	+ ·081	− ·123	·000	+ ·042	− ·079
2·10	+ ·120	− ·040	− ·062	+ ·059	− ·119	+ ·001	+ ·046	− ·092
2·24	+ ·089	+ ·109	+ ·102	+ ·066	+ ·043	+ ·035	+ ·168	− ·147
1·00	+ ·040	+ ·049	+ ·056	+ ·030	+ ·019	+ ·016	+ ·075	− ·066

TABLE VII.

Rates of Change in the Harmonics Collected.

(The unit is 0·001 in log N_0.)

Zone.	sin θ.	cos θ.	sin 2θ.	cos 2θ.	sin 3θ.	cos 3θ.	sin 4θ.	cos 4θ.
−1°	+40	+49	+56	+30	+19	+16	+75	−66
−3	−38	+33	−53	+19	−50	+ 3	+23	−29
−4	+29	+28	+11	+54	−47	+18	+52	−11
−5	−25	+36	−33	+ 8	−33	+17	+43	− 7
−6	−18	+42	−12	+10	+24	+24	+ 8	−24
Mean	− 2	+38	− 6	+24	−17	+16	+40	−27

fourth harmonic shown in Table VI. (and confirmed by zones −15° to −19° and even further south, as may be seen on reference to Table VI. on p. 530 of *M.N.*, 81) may be significant.

Summary.

Counts for the two San Fernando zones −4° and −6° are analysed by the standard process: and counts previously given for Algiers −1° are also analysed. With similar results for −3° and −5° the mean first harmonic for the whole belt −1° to −6° is seen to differ widely from that for −15° to −19°. Counts of zones near −10° would thus be specially valuable for investigating the nature of the relationship.

Mean Areas and Heliographic Latitudes of Sunspots in the Year 1923.

(*Communicated by the Astronomer Royal.*)

The results here given are in continuation of those in the *Monthly Notices*, 84, 30, and are derived from the measurement of photographs taken at the Royal Observatories of Greenwich and the Cape, and in India at Kodaikánal Observatory and at Dehra Dûn.

Table I. gives the mean daily area of umbræ, whole spots, and faculæ for each synodic rotation of the Sun for the year 1923, and Table II. gives the same particulars for each entire year from 1913 to 1923 inclusive. The areas are given in two forms: firstly, projected areas, *i.e.* as seen and measured on the photographs, these being expressed in millionths of the Sun's apparent disc; secondly, areas corrected for foreshortening, the areas in this case being expressed in millionths of the Sun's visible hemisphere.

Table III. gives for each rotation in the year 1923 the mean daily area of the whole spots (corrected for foreshortening) and the mean heliographic latitude of the spotted areas for spots north and for spots south of the equator, together with the mean heliographic latitude of the entire spotted area and the mean distance from the equator of all spots. Table IV. gives the same information for each year as a whole from 1913 to 1923. Tables II. and IV. are thus in continuation of similar tables for the years 1874 to 1888 on pp. 381 and 382 of vol. xlix. of the *Monthly Notices*; for the years 1889 to 1902 on pp. 465 and 466 of vol. lxiii.; and for the years 1901 to 1914 on pp. 402 and 403 of vol. lxxvi.

The rotations in Tables I. and III. are numbered in continuation of Carrington's series (*Observations of Solar Spots made at Redhill*, by R. C. Carrington, F.R.S.), No. 1 being the rotation commencing 1853 November 9. The assumed prime meridian is that which passed through the ascending node at mean noon of 1854 January 1, and the assumed period of the Sun's sidereal rotation is 25·38 days.

The principal features of the record for the year 1923 were as follows:—

1. A well-marked decrease in the mean daily area of spots—55 units as compared with 252 units for the previous year and 1537 units for the maximum year of the cycle.

2. A decrease of about 50% in the amount of faculæ as compared with the previous year.

3. The mean solar latitude of all spots was 15°. Spots in high latitudes belonging to the new cycle were increasingly frequent during the latter half of the year. Dividing the spots according to their latitude and the Mt. Wilson magnetic classification into old and new cycle spots, the mean latitudes are respectively $6\frac{1}{2}°$ and $24\frac{1}{2}°$.

4. There were 171 spotless days as compared with 109 in the previous year.

5. The number of separate groups lasting for at least two days was 33 as compared with 76 for the year 1922. Of these 33 groups, 19 were in the northern hemisphere and 14 in the southern. Besides these, there were recorded 30 small groups (mostly consisting of one or two very small spots) visible for one day only; 20 were in the northern and 10 in the southern hemisphere.

6. The year 1923 showed, therefore, the usual characteristics of a minimum year, viz. slight spot and facula activity: occasional small groups of spots appearing near the solar equator as representatives of the passing cycle, with increasing activity in the higher latitude zones.

TABLE I.

No. of Rotation.	Rotation commenced G.M.T. (Civil).		No. of Days on which Sun photographed.	Mean of Daily Areas. Projected.			Corrected for Foreshortening.		
				Umbræ.	Whole Spots.	Faculæ.	Umbræ.	Whole Spots.	Faculæ.
	1923.	d.							
927	Jan.	4·86	27	2	8	314	2	8	351
928	Feb.	1·20	28	1	3	176	1	2	216
929	Feb.	28·54	27	3	9	141	2	7	163
930	Mar.	27·85	27	15	58	145	9	34	163
931	April	24·12	27	5	19	148	4	13	171
932	May	21·35	28	7	30	131	5	24	161
933	June	17·55	27	31	168	142	19	104	193
934	July	14·75	27	3	10	305	2	7	334
935	Aug.	10·97	27	14	60	115	9	39	142
936	Sept.	7·21	27	32	178	162	23	129	204
937	Oct.	4·49	28	30	154	218	21	112	253
938	Oct.	31·78	27	27	159	233	20	125	272
939	Nov.	28·09	27	8	40	137	5	25	166

TABLE II.

Year.	No. of Days on which Sun photographed.	Mean of Daily Areas. Projected.			Corrected for Foreshortening.		
		Umbræ.	Whole Spots.	Faculæ.	Umbræ.	Whole Spots.	Faculæ.
1913	365	2	10	87	1	7	95
1914	365	40	195	396	31	152	454
1915	365	186	920	1289	137	697	1521
1916	365	186	978	1512	135	724	1785
1917	365	346	2101	1882	247	1537	2305
1918	364	255	1504	1526	188	1118	1882
1919	365	253	1437	1445	182	1052	1729
1920	366	156	837	1052	112	618	1219
1921	365	110	571	647	79	420	739
1922	365	63	346	358	45	252	415
1923	365	15	74	189	10	55	222

TABLE III.

No. of Rotation.	Rotation commenced G.M.T. (Civil).		Spots NORTH of the Equator.		Spots SOUTH of the Equator.		Mean Heliographic Latitude of entire Spotted Area.	Mean Distance from Equator of all Spots.
			Mean of Daily Areas.	Mean Heliographic Latitude.	Mean of Daily Areas.	Mean Heliographic Latitude.		
	1923	d.		°		°	°	°
927	Jan.	4·86	7	5·80	1	1·77	+4·76	5·25
928	Feb.	1·20	1	11·23	0·4	7·00	+7·29	10·31
929	Feb.	28·54	6	6·06	0·3	26·22	+4·45	7·07
930	Mar.	27·85	19	4·91	16	5·91	−0·05	5·37
931	Apr.	24·12	2	7·05	11	5·40	−3·32	5·68
932	May	21·35	16	9·70	8	8·84	+3·47	9·42
933	June	17·55	95	8·06	9	3·20	+7·08	7·64
934	July	14·75	4	5·91	3	25·92	−7·66	14·44
935	Aug.	10·97	1	20·59	38	27·60	−26·82	27·49
936	Sept.	7·21	46	21·56	84	19·34	−4·89	20·12
937	Oct.	4·49	34	3·66	78	11·23	−6·74	8·95
938	Oct.	31·78	81	29·60	43	27·50	+9·74	28·87
939	Nov.	28·09	24	28·30	1	22·36	+27·23	28·17

TABLE IV.

Year.	No. of Days on which Sun photographed.	Spots NORTH of the Equator.		Spots SOUTH of the Equator.		Mean Heliographic Latitude of entire Spotted Area.	Mean Distance from Equator of all Spots.
		Mean of Daily Areas.	Mean Heliographic Latitude.	Mean of Daily Areas.	Mean Heliographic Latitude.		
			°		°	°	°
1913	365	5	24·81	2½	20·08	+9·81	23·23
1914	365	99	21·11	53	23·06	+5·71	21·79
1915	365	379	17·95	318	19·75	+0·77	18·77
1916	365	470	15·14	254	17·05	+3·83	15·81
1917	365	860	13·82	677	15·66	+0·84	14·63
1918	364	609	11·97	509	13·69	+0·29	12·75
1919	365	559	9·85	493	11·79	−0·28	10·76
1920	366	208	11·31	410	9·98	−2·83	10·43
1921	365	261	7·27	159	8·92	+1·13	7·90
1922	365	161	8·82	91	6·62	+3·23	8·02
1923	365	33	13·88	22	17·32	+1·34	15·26

Royal Observatory, Greenwich:
1924 *July* 16.

A New Variable Star in Gemini. By A. Stanley Williams.

R.A. = 7 h 47·9 m; Decl. = + 32° 50′ (1855).

The variability of this star was first noted in 1921 November, when comparing two photographs taken about twenty years previously with a 4·4-inch portrait lens. Visual observations with a 6½-inch reflector were commenced in the same year, and these soon confirmed the variability of the star, but showed that the range of variation is small. Unfortunately, Gemini is one of the winter constellations, when the work of observation becomes more difficult, so that the observations so far secured are not so numerous as might be wished. However, as they appear to indicate the nature and period of the variation, it seems desirable that the results should now be published.

Particulars of the comparison stars and the light-scale are given in Table I. The places of b and e, and that of the variable itself as stated above, are very rough, but should suffice for identification. The star e is the brighter and preceding of two stars. The magnitudes in the last column of the table are based on the assumptions that b is 10·0 mag. and that the value of a "step" is 0m·05 mag.

TABLE I.

Star.	Name.	R.A.	Decl.	B.D. Mag.	Light-Scale.	Adopted Mag.
	°	h m	° ′			
d	B.D. 32 1651	7 48·4	+32 52	9·5	19·4	9·71
a	B.D. 32 1649	7 47·6	+32 42	9·5	18·2	9·77
b	...	7 47·4	+32 47	...	13·6	10·00
e	...	7 46·5	+32 48	...	3·7	10·50

The total number of good observations is 47, after excluding those that were more or less probably affected by cloud or haze. They are pretty equably distributed over the three winters of 1921–2, 1922–3, and 1923–4. The adopted elements of variation are:

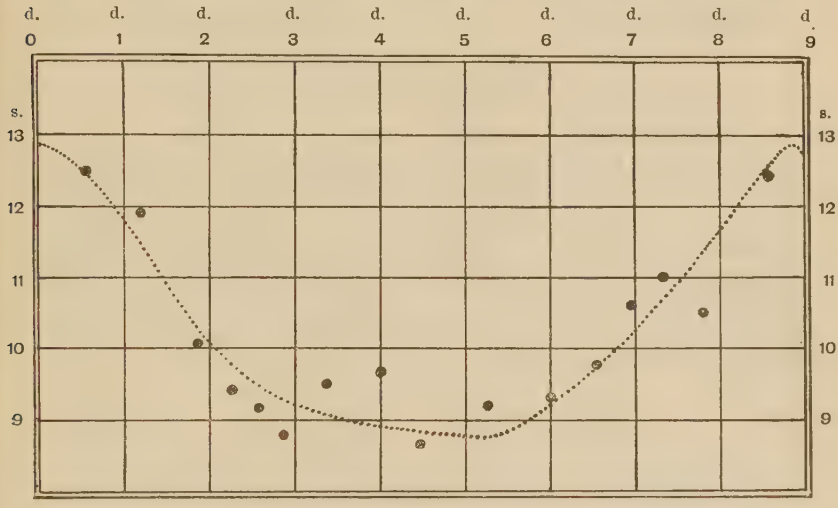
Maximum = J.D. 2423106d·5 + 8d·875 E.

Using these elements the observations have been formed into 16 groups or mean observations, with 5 observations to a group. But in order to increase the number of groups there is some overlapping, each group consisting of three fresh observations and two taken from the preceding group. Table II. contains these grouped or mean observations, and they are also shown in diagram form in the adjoining figure.

TABLE II.

Phase.	Brightness.	Phase.	Brightness.	Phase.	Brightness.	Phase.	Brightness.
d	s	d	s	d	s	d	s
0·55	12·5	2·58	9·2	4·45	8·7	6·94	10·6
1·19	11·9	2·87	8·8	5·25	9·2	7·33	11·0
1·85	10·1	3·35	9·5	6·01	9·3	7·82	10·5
2·23	9·4	4·00	9·7	6·55	9·8	8·56	12·4

The mean light-curve as drawn in the diagram shows that the variable is probably a Cepheid, but that the decrease in brightness is nearly as quick as the increase, and the possibility of the minimum being actually midway between two maxima is not absolutely excluded.



Light-curve of variable star in Gemini.

The small number of observations and the slight range in brightness render this point uncertain at present. The range in brightness according to the mean light-curve is in fact only from 12·85 steps to 8·85, corresponding to 0·2 magnitude. This small range of no more than a fifth of a magnitude makes the star difficult to observe visually by the method of Argelander, and it is desirable that it should be observed with some form of photometer.

It should be mentioned that 3 out of the 47 single observations were rejected for discordance. They all three make the star too faint. It is quite likely that these observations may have been affected by cloud, which is always to be feared in a climate like that of Cornwall. The writer has frequently seen to form in the course of a few seconds of time, and in a perfectly clear sky, a patch of misty cloud or haze covering only a few square degrees, which cloud after persisting for a

few minutes would gradually disappear. Such a patch of cloud or haze might easily escape the notice of an observer in the night time, particularly a short-sighted one working without spectacles, as it might only produce a slight dimming of the fainter stars.

Summary.

(1) The star is a short-period variable, probably a Cepheid.
(2) Period . . . . . 8·875 days.
(3) Epoch of maximum . . . . J.D. 2423106·5
(4) Maximum brightness . . . . . 10·04 mag.
(5) Minimum brightness . . . . . 10·24 ,,
(6) Range . . . . . 0·20 ,,

Polvarth, St. Mawes:
1924 *July* 21.

Elements of Variation of the Variable Star 7·1922 Geminorum.

By A. Stanley Williams.

Some early photographic observations of this variable, R.A. = $7^h\ 07^m{\cdot}7$, Decl. = $+26°\ 15'$ (1855), were published in the *Monthly Notices*, 82, 416. Three maxima have now been observed visually with a $6\frac{1}{2}$-inch reflector, and by making some reasonable assumptions, the rough times of three others have been inferred from the early photographic observations. Particulars of these are given below:

E.	Observed Maximum. Date.	J.D.	Computed Max.	O−C.	Brightness.
				d	
−25	1900 May 9	2415149±	5142	+ 7	...
−24	1901 Mar. 29	2415473±	5463	+10	...
−23	1902 Feb. 15	2415796±	5784	+12	...
0	1922 Apr. 20	2423165	3167	− 2	10 mag.
1	1923 Mar. 11	3490	3488	+ 2	10 ,,
2	1924 Jan. 26	3811	3809	+ 2	9 ,,

The period derived from the three visual observations is 322 days, but carried back this does not satisfy the three photographic observations, which seem to indicate a rather shorter period. The adopted provisional elements of variation are therefore:

$$\text{Maximum} = \text{J.D. } 2423167 + 321^d\ \text{E.}$$

Polvarth, St. Mawes:
1924 *July* 24.

The Determination of the Curvature of Space-Time in de Sitter's World.
By Knut Lundmark, Ph.D.

1. In a very interesting paper in *M.N.*, **84**, 363, 1924, and in several letters to the editor of *Nature*,* Dr. Silberstein has derived from de Sitter's cosmology a formula for the Doppler shift, $\frac{d\lambda}{\lambda}$, of a star in inertial relative motion to the observer.

The formula may be written

$$\frac{d\lambda}{\lambda} = \left(1 - \frac{v_0^2}{c^2}\right)^{-\frac{1}{2}} \left\{1 \pm \left[1 - \cos^2 \frac{r}{R} \Big/ \left(1 - \frac{v_0^2}{c^2}\right)\right]^{\frac{1}{2}}\right\} - 1,$$

where v_0 = the individual velocity of a star at a past or future epoch,
c = velocity of light,
r = distance of the star from observer,
R = the curvature radius of space-time.

The expression $\left(1 - \frac{v_0^2}{c^2}\right)^{-\frac{1}{2}}$ as suggested by astronomical observations will be $\leq 1{\cdot}0000444$, and for most of the stars $\leq 1{\cdot}000001$. Taking account of this fact, the formula can be simplified to

$$\frac{d\lambda}{\lambda} = \pm \sin \frac{r}{R},$$

or for small or moderate values of $\frac{r}{R}$,

$$\frac{d\lambda}{\lambda} = \pm \frac{r}{R}.$$

Dr. Silberstein makes use of this formula for computing R by using a few of the observed radial velocities for globular clusters. Before entering on a discussion concerning the value of R thus derived, we will for a moment look into the matter of the nature of the spectral shifts observed for globular clusters and spiral nebulæ, as it may not be without interest to know if the spectral displacements are of the Doppler nature.

2. The velocities for these objects are mainly due to the wonderful spectrographic work performed at the Lowell Observatory by Dr. V. M. Slipher. It is well known that the velocities are very high and in many respects of similar nature for the two classes of objects, except that the spiral nebulæ show a very large spectral shift towards the red. Some astronomers seem to have been rather sceptical as to the meaning of the spectral displacements observed, and several times the question has been raised: Are the displacements observed of the Doppler character? *i.e.* is $\frac{d\lambda}{\lambda}$ constant within the limits of the errors of measurement? This is

* *Nature*, 1924 March 8, April 26, and June 7.

certainly the case, and in illustration we may examine the following measures of a spectrogram of the Andromeda Nebula, which I am allowed to quote through the courtesy of Professor Wright at the Lick Observatory.

λ.	$\frac{\Delta\lambda}{\lambda}$.	λ.	$\frac{\Delta\lambda}{\lambda}$.
$\mu\mu$		$\mu\mu$.	
486	0·0012	427	0·0011
473	10	423	10
455	11	410	14
438	11	405	6
438	16	405	11
434	15	403	14
433	13	398	5
429	0·0017	397	0·0010

The mean is $0{\cdot}00116 \pm 0{\cdot}00008$, and the shift is evidently a Doppler one. The same applies to the velocities of the globular clusters.

3. Another question is, whether such a large Doppler shift represents motion in the line of sight alone or is caused in other ways? The validity of the Doppler principle has been proved by laboratory experiments only for velocities smaller than 1 km./sec. or so. The measures of stellar spectrograms giving such velocities as can be computed from the laws of gravitational astronomy (*e.g.* velocity in orbit of *Venus*, *Earth*, and *Mars*, radial velocities of comets) have proved the correctness of the Doppler formula for velocities as high as 100 km./sec., and thus it seems allowable to assume that the displacements found for globular clusters and spiral nebulæ are due to motions of the objects in the non-relativistic sense or to motions and the above-mentioned effect of the curvature of space-time.

If we treat the 18 velocities known for globular clusters and the 43 for spiral nebulæ as real motions, and compute the apex for the *Sun's* motion relatively to the two systems by solving equations of the form

$$X \cos\alpha \cos\delta + Y \sin\alpha \cos\delta + Z \sin\delta + K - V = 0,$$

where K is a systematic correction to the velocities and V the observed radial velocity, we obtain the results:

Globular Clusters.		Spiral Nebulæ.
$A = 20^h{\cdot}4$	$A = 21^h{\cdot}0$	$A = 20^h{\cdot}3 \pm 3^h{\cdot}0$ (m.e.)
$D = +60°$	$D = +64°$	$D = +75° \pm 30°$
$V_0 = 305$ km./sec.	$V_0 = 304$ km./sec.	$V_0 = 651 \pm 135$ km./sec.
$K = +31$ km./sec.	$K = 0$ (assumed) *	$K = +793 \pm 88$ km./sec.

* The mean errors of the solution cannot be given here, but we may quote the mean errors derived at a solution using 16 globular clusters (*Publ. A.S.P.*, 35, 318, 1923), from which we find $A = 21^h{\cdot}3 \pm 6^h{\cdot}0$, $D = +64° \pm 22°$, $V_0 = 266 \pm 112$ km./sec., and $K = 0$.

Thus the *Sun* has a different motion relatively to the systems of globular clusters and spiral nebulæ from that relative to the nearest stars ($A = 18^h$, $D = +30°$, $V_0 = 20$ km./sec.). The explanation may simply be that *our local system as a whole* has the motion found above relatively to the huge systems of globulars and spirals. When using the nearest stars for a determination of the apex we do not get this motion, but the motion of our *Sun* relative to the local system. If we include in our solution the nearest stars as well as more distant objects, we obtain a value for the apex and the velocity V_0 intermediate between the values obtained by using separately very close and remote objects. This is in harmony with the fact* that Dyson and Thackeray obtained a progression in the position of the solar apex when grouping the Groombridge stars according to apparent magnitude; that is to say, when going from nearer to more distant stars. The progression in question shows a change of the position of the apex towards higher declinations, and increases the right ascension and the value of V_0.†

It is of interest to note that our motion relatively to the spirals is directed towards galactic longitude $75° \pm 17°$, or nearly at right angles to the direction (gal. long. $= 325° - 335°$) where many modern researches place the centre of our galactic system. This fact might suggest that our local cluster is revolving around this centre or moving in an orbit with a period of roughly $3 \cdot 10^9$ years.

A support for the view that the systematic motions of the spiral nebulæ and globular clusters derived above might be a real phenomenon not showing any effect of the curvature of space-time is perhaps obtained from using the proper motions of the spiral nebulæ. The best values we have for these are undoubtedly the ones derived by van Maanen from measuring 882 "nebular points" in 7 spirals.‡ Using the values for μ_α and μ_δ according to Dr. Smart's re-discussion,§ and solving Airy's equations,

$$x \sin \alpha - y \cos \delta = \mu_\alpha \cos \delta$$
$$x \cos \alpha \sin \delta + y \sin \alpha \sin \delta - z \cos \delta = \mu_\delta,$$

we find the following value for the apex:

$$A = 1^h{\cdot}7 \pm 4^h{\cdot}7 \text{ (m.e.)}$$
$$D = +63° \pm 45°$$
$$g = +0''{\cdot}0074 \pm 0''{\cdot}0055.$$

Of course considerable uncertainty is involved in this result on account of the scantiness of the material at hand. Still it may be more

* *M.N.*, 65, 428, 1905.

† *Kungl. Svenska Vetenskapsakademiens Handlingar*, Bd. 60, No. 8, 1920.

‡ The proper motions derived by Curtis for 66 spirals and by the present writer for 82 spirals, to be mentioned later in this paper, are not considered to be real by either author. The values obtained give an upper limit for the size of the mean proper motion, and thus are useful as also giving an upper limit for the size of the mean parallax.

§ *M.N.*, 84, 333, 1924.

than a chance that we obtain the same direction for the apex from the proper motions as from the radial velocities. As the former are not affected by a curvature of space-time, the conclusion would be that the spectral shifts observed for globular clusters are due to real motion and show no influence of the world-curvature. On account of the uncertain material we leave the question open until more information is accumulated.

The apex computed from the radial velocities of spiral nebulæ seems to be fairly well determined, inasmuch as calculations performed for a few years ago only, using 15 or 18 objects, gave essentially the same results.*

4. Turning to the question of the determination of the curvature of space-time from the radial velocities of the globular clusters, it may at first be noticed that there is an essential difference between the views held by Weyl and Eddington and the ones held by Silberstein. According to the former authors there are no terms in the expression for the Doppler shift which will explain approaching velocities, whereas according to Dr. Silberstein both plus and minus occur in the formula, which thus accounts as well for receding as for approaching velocities. As Dr. Silberstein promises to discuss this question at length in a coming edition of his book, *The Theory of Relativity*, we may take his views for granted, and see what result can be obtained by using the material at hand for a determination of R.

Dr. Silberstein has not given, and will probably not be able to give, any justification for the use of the velocities of the globular clusters for a determination of R. These objects are probably among the most distant celestial objects we know at present, but how do we know that they are so far away that the effect of the curvature of space-time outweighs the effect of the real motions of the clusters themselves? In fact, the smallness of the K term for the globular clusters, found above to be +31 km./sec., while for the spiral nebulæ this term is nearly +800 km./sec., suggests that the former objects are comparatively near as compared with the latter, and that the globulars are little if at all affected by the slowing down of atomic vibrations in distant objects in de Sitter's world which might be erroneously interpreted as a motion of recession.

But granted the globular clusters are distant enough to permit a determination of the R, it does not seem possible at present to get an accurate determination from the material at hand. It is to be regretted that Dr. Silberstein in his articles has not used the whole material. Of the 16 velocities so far published he uses only 7. He seems to have been aware of the existence of at least 10 cluster velocities, but he says: "In view of the said large p.e. of the latter (the velocities of clusters), only those are worth considering the radial velocities of which are not much less than a hundred km./sec." Of course, there is no good reason for selecting such an arbitrary limit for excluding objects *which do not give a rather constant value of the R.* Furthermore, the velocities of globular clusters have certainly a much smaller mean

* *Kungl. Svenska Vetenskapsakademiens Handlingar*, Bd. 60, No. 8, 1920; *Astr. Nachr.*, 209, 369, 1919.

error than 100 km./sec. Taking the two objects measured both at the Lowell Observatory and at Mount Wilson, we have :

	Sanford.	Slipher.
N.G.C. 5024	−200 km./sec.	−175 km./sec.
5272	−140	−125

indicating a rather small probable error.

5. As the spectrograms of the spiral nebulæ are of much the same nature as those of the globular clusters, we may obtain an idea concerning the uncertainty in deriving cluster or nebular velocities from the following small table giving the results of several different determinations of the radial velocities of two spiral nebulæ :

N.G.C. 224.		N.G.C. 1068.	
Radial Velocity. km./sec.	Authority.	Radial Velocity. km./sec.	Authority.
−329	Pease and Adams.	+910	Moore.
−297	,, ,,	+940	Lundmark.*
−300	Slipher.	+765	Pease.
−350	Wolf.	+1060	Slipher.
−304	Wright.	+1150	,,
		+1080	,,
−316 ± 10 (m.e.)		+1135	,,
		1005 ± 53 (m.e.)	

In the case of N.G.C. 1068 there seems to be a rather high mean error in the radial velocity, but the peculiarities of the spectrum suggest that the differences found by different observers may be partly real.

The mean error in the velocities of globular clusters and spiral nebulæ does not seem to be higher than 25–30 km./sec. Of course, it is not justifiable to exclude the values of velocities even below that limit in computing R, as the mean of the small velocities may very well have a real significance, and thus give an important contribution to the determination of R. The very fact that small radial velocities exist among the distant clusters shows that the R must be greater than the value found by Dr. Silberstein from his *selected* values of existing radial velocities.

Using the 16 velocities of globular clusters together with two as yet unpublished values, kindly communicated in a letter by Dr. Slipher, we find the values for R given in Table I. For the distances of the globular clusters values are used derived by Shapley and by the present writer using different methods.† It is clear from an inspection of the table that the rather large dispersion in R found by us cannot be explained by the uncertainty in the relative parallaxes of the clusters.

* Remeasured on the request of Dr. Moore.

† *Mount Wilson Contr.*, pp. 151–157, 1918; *Astrophys. Journ.*, 48-49, 1918; *Kungl. Svenska Vetenskapsakademiens Handl.*, Bd. 60, No. 8, 1920.

TABLE I.

The Curvature Radius of Space-Time as derived from Globular Clusters.

Object.	Radial Velocity.	Distance in Parsec.		R in km.	
		Acc. to Shapley.	Acc. to Lundmark.	From Shapley's Parallaxes.	From Lundmark's Parallaxes.
N.G.C. 1851	+300	$1{\cdot}72 \cdot 10^4$	$1{\cdot}41 \cdot 10^4$	$3{\cdot}6 \cdot 10^{12}$	$2{\cdot}9 \cdot 10^{12}$
1904	+300	2·56	2·50	5·3	5·2
5024	−190	1·89	1·49	6·2	4·8
5272	−135	1·39	0·80	6·4	3·7
5904	+ 10	1·25	0·75	77·3	46·4
6093	+ 70	2·00	1·41	17·7	12·5
6205	−300	1·11	0·67	2·3	1·4
6218	+160	1·23	0·89	4·8	3·4
6229	−100	4·35	7·14	26·9	44·3
6266	− 50	1·52	1·43	18·8	17·7
6273	+ 30	1·59	1·56	32·8	32·2
6333	+225	2·50	2·27	6·9	6·2
6341	−160	1·23	1·03	4·8	4·0
6626	0	1·85	1·92	∞	∞
6934	−350	3·33	3·70	5·9	6·5
7078	− 95	1·47	1·19	9·6	7·7
7089	− 10	1·56	1·01	96·5	62·5
7099	−125	1·72	1·30	8·5	6·4
Small Magellanic Cloud	+168	3·10 *	...	11·4	...
Great Magellanic Cloud	+278	3·45	2·78 †	7·7	6·2

The graph shows that there is little or no correlation between V and r, in contrariety to what is to be expected from the theory of Dr. Silberstein. Computing the correlation coefficient, r, we find the following value:

$$r = 0{\cdot}255 \pm 0{\cdot}227,$$

which cannot be considered to have a real significance.

The parallaxes for globular clusters derived by Shapley and by the writer involve the supposition that the apparent diameters of the clusters are very intimately related to the parallaxes or distances, and if Dr. Silberstein's theory is correct, they also are closely correlated to the apparent diameters. Computing the correlation between the apparent diameters and radial velocities, we find:

$$r = -0{\cdot}141 \pm 0{\cdot}223 \text{ (the writer's diameters).}$$
$$r = -0{\cdot}185 \pm 0{\cdot}229 \text{ (Shapley's diameters).}$$

As the dispersion in R is $26{\cdot}1 \cdot 10^{12}$ km. and thus considerably higher than what could be expected from the dispersions in V and r,

* *Harv. Circ.*, No. 255, 1924. † *Observatory*, 1924 Sept.

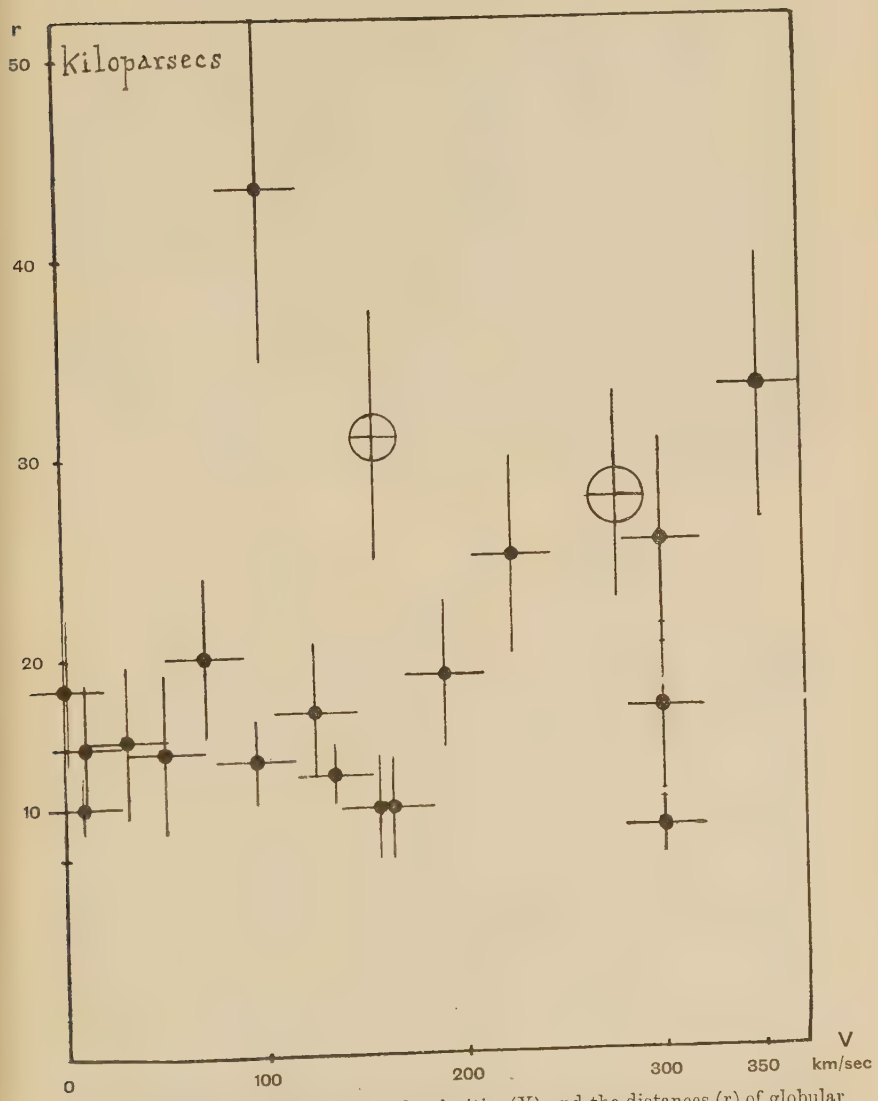


FIG. 1.—Relation between the radial velocities (V) and the distances (r) of globular clusters. The values for the Magellanic clouds have also been inserted, and are denoted by circles. The crosses indicate the size of the estimated probable error in either direction. This diagram, as giving all the known cases, supersedes that of Dr. Silberstein in *Nature*, 1924 June 7.

it does not seem that the curvature of space-time, at least for the present, can be determined with any accuracy by using the displacements in the spectra of globular clusters.

But if we assume that a value for R can be obtained, it will certainly be considerably larger than that of $6 . 10^{12}$ km. adopted by Silberstein, and which he considers worthy of confidence. From Table I. the mean value is found to be $19{\cdot}7 . 10^{12}$ km., using Shapley's distances, which should be preferred to the writer's on account of the more extensive material on which they are based. We have excluded N.G.C. 6626, as no stress should be laid on the fact that this cluster gives infinity for the curvature radius. The actual value may well be $80 . 10^{12}$ km. or higher, but the very small velocity of the object favours a higher value for R than that adopted by Dr. Silberstein.

6. Dr. Silberstein applies his R to the problem of determining at what distance from the *Sun*, r^*, de Sitter's "repulsion" is just balanced by the gravitational attraction. He computes the r^* at 3·45 parsecs, and thinks there are several stars close to this "neutral" sphere. As this result, if correct, is of an unusual importance for our conceptions of the stellar system, we will see if the result can be confirmed.

TABLE II.

The Stars having a Parallax $\geq 0''{\cdot}20$.

	Name.	α (1900).	δ (1900).	π Abs. Red.
		h m	° ′	″
1	Groombr. 34 *	0 12·7	+43 27	0·282
2	[+55°·100	26·6	+55 27	0·200]†
3	v. Maanen's star	43·9	+ 4 55	0·255
4	τ Ceti	1 39·4	−16 28	0·324
5	Fox's star	2 50·3	+52 5	0·254
6	ϵ Eridani	3 28·2	− 9 48	0·318
7	o^2 Eridani *	4 10·7	− 7 49	0·198
8	Cord. Z.V. 243	5 7·7	−44 59	0·309
9	Sirius *	6 40·7	−16 35	0·374
10	Procyon *	7 34·1	+ 5 29	0·313
11	Groombr. 1618	10 5·3	+49 58	0·208
12	W.B. $10^h{\cdot}234$	14·2	+20 22	0·197
13	22 H Camelop.	57·9	+36 38	0·409
14	Innes' star	11 12·0	−57 2	0·337
15	AOe 11677	14·8	+66 23	0·215
16	A.G. Berl. 5003	13 40·7	+15 26	0·216

* Double star or physical system.
† Excluded, as the parallax found by Peters needs confirmation.

TABLE II.—*continued.*

	Name.	α (1900).	δ (1900).	π Abs. Red.
		h m	° ′	″
17	Proxima Cent.	14 22·9	−62 15	0·761
18	α Centauri *	32·8	−60 25	0·763
19	AOe 17415	17 37·0	+68 26	0·238
20	Barnard's star	52·9	+ 4 25	0·555
21	70 Ophiuchi *	18 0·4	+ 2 31	0·196
22	Σ 2398 *	41·7	+59 29	0·292
23	Altair	19 45·9	+ 8 36	0·214
24	δ Pavonis	58·9	−66 26	0·203 †
25	279 G Sagittarii *	20 4·6	−36 21	0·209
26	61 Cygni *	21 2·4	+38 15	0·310
27	Lac. 8760	11·4	−39 15	0·233
28	ε Indi	55·7	−57 12	0·269
29	Krüger 60 *	22 24·4	+57 12	0·256
30	Lac. 9352	59·4	−36 26	0·282
31	Cord. 32416	23 59·5	−37 51	0·219

The absolute parallaxes have been used, and van Maanen's reductions for systematic errors have been adopted.

At first it will be noticed that there are not, as Dr. Silberstein states, 19 stars having a parallax $\geq 0''{\cdot}20$, but 30 or 31, as will be seen from the accompanying Table II. Our knowledge concerning the southern sky is as yet incomplete, and, besides, if we had a complete survey of radial velocities we certainly should find several stars with large parallaxes among those having high radial motions and moderate proper motions. Anyhow, the stars known at present within a radius of 5 parsecs do not show such a crowding as required by Dr. Silberstein. Not counting components of double stars or distant companions except *Proxima Centauri*, we find that there are 12 stars between the limits $0''{\cdot}255$ and $0''{\cdot}340$, or 3·92 and 2·94 parsecs. If the 30 stars filled uniformly a sphere of 5·10 parsecs, the number of stars which would fall within these limits would be 8·0. Thus there is scarcely any crowding that cannot just as well be accounted for by our imperfect knowledge of the parallaxes of the stars within this sphere. If we assume, which is very reasonable, that the unknown stars falling within this sphere are more likely to have a parallax smaller than $0''{\cdot}255$ than larger, we see that the very slight tendency to a crowding at present suggested may easily be obliterated.

As from the complete material of radial velocities for globular clusters a much higher value for R is suggested than that derived by Silberstein, we find that the value for r^* should be 11·29 parsecs, which corresponds to a parallax of $0''{\cdot}091$. In order to see if there is any

* Double star or physical system.

† Included on the strength of the spectral and proper-motion parallax.

crowding of stars around this value, we have grouped the known parallaxes according to the following table:—

Parallax.	Number of Trigonometric Parallaxes.	Number of Spectroscopic Parallaxes.	$A(\pi)$.	$A'(\pi)$.
$0''.191 - 0''.200$	2	2	2	2
$0''.181 - 0''.190$	4	3	6	5
$0''.171 - 0''.180$	5	2	11	7
$0''.161 - 0''.170$	5	0	16	7
$0''.151 - 0''.160$	2	7	18	14
$0''.141 - 0''.150$	8	4	26	18
$0''.131 - 0''.140$	8	7	34	25
$0''.121 - 0''.130$	5	7	39	32
$0''.111 - 0''.120$	9	11	48	43
$0''.101 - 0''.110$	14	17	62	60
$0''.091 - 0''.100$	17	33	79	93
$0''.081 - 0''.090$	19	24	98	117
$0''.071 - 0''.080$	39	37	137	154
$0''.061 - 0''.070$	60	62	197	216
$0''.051 - 0''.060$	84	129	281	345

The material used is obtained for the trigonometric parallaxes from our card catalogue containing about 1750 stars measured for parallax, and for the spectroscopic parallaxes from the catalogues of Adams (and co-workers), Rimmer, Abetti, Shapley (and co-workers), Edwards, Lundmark and Luyten, which altogether contain nearly 3000 stars. Absolute mean parallax-values have been used.

The figures $A(\pi)$ and $A'(\pi)$ giving the number of stars having a parallax larger than π can easily be used for computing the number of stars to be expected within the different shells if the stars are uniformly distributed within the sphere having a radius of 20 parsecs.

There does not seem to be any crowding around the value $\pi = 0''.091$, and although the parallaxes $\geq 0''.20$ are certainly more completely known than are the parallaxes between $0''.20$ and $0''.10$, still the fairly regular distribution of the known parallaxes does not suggest that further scrutiny will detect such a crowding as expected on Dr. Silberstein's theory.

7. As the value of R derived from the globular cluster material is rather uncertain, we will examine other classes of distant objects where it might be possible to find an effect of the curvature of space-time. Such classes of rather distant objects are the Cepheids, the Novæ, the O stars, the eclipsing variables, certain classes of red stars, and the spiral nebulæ. The distances of these objects are generally sufficiently large for the curvature of space-time to be perceptible if it exists at the

distances of the globular clusters. Besides, there ought to be a progres sion in the numerical values of the radial velocities with the distance if we have a group of bodies with a considerable range in the individual distances, such as the Cepheids.

(*a*) *Cepheids.* — The distances of these stars, first derived by Hertzsprung and later revised by Shapley by using the data for proper motion in connection with the period-luminosity law, are the foundation for the adopted distances of globular clusters. As the radial velocities have been used by deriving the mean parallax without applying any correction for a possible influence of the curvature of the space-time, it may not be allowable to compute the R from the distances thus derived.* Of course the same remark refers to the use of the distances of the globular clusters, as they are founded on the mean parallax of Cepheids.

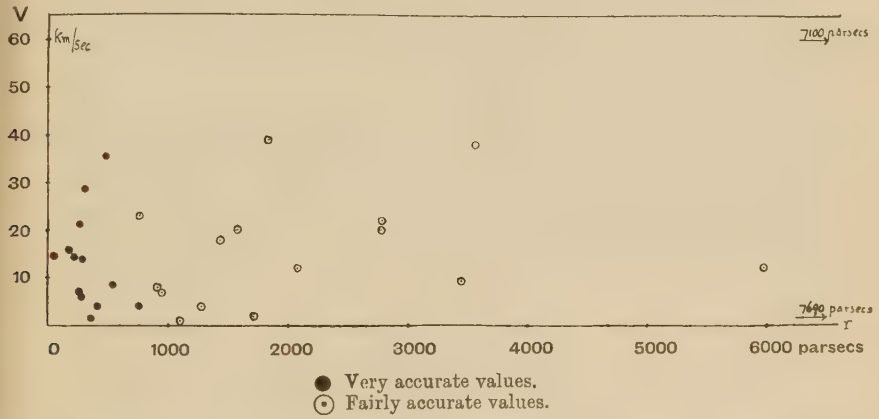


FIG. 2.—Radial velocities of Cepheids plotted against their distances. Two stars fall outside the diagram, as indicated by the arrows.

Now, the direct measured parallaxes for Cepheids seem to confirm the order of magnitude of the distance, and, besides, the influence of the curvature of space-time in the radial velocities is probably not very large when we deal with objects comparatively near. Thus we think we are justified in using the Cepheids for a determination of R.

For the 94 Cepheids having a period longer than one day the mean distance computed from the catalogue of Shapley is found to be 1940 parsecs.† Plotting the radial velocities known and the corresponding distances, we see that no progression is indicated in the former, as ought to be the case if R is a constant.

* Strictly speaking, the mean parallax of the Cepheids was derived by using the τ and ν components of their proper motions compared with the value of the velocity relative to the stellar system, or V_0, derived from other classes of stars. But Shapley has shown that in the radial velocities of Cepheids there is no systematic motion left. A computation of the apex of the Cepheids using the known radial velocities gives a value for V_0 in substantial agreement with the one adopted by Shapley.

† *Mount Wilson Contr.*, 153, 1918; *Astrophys. Journ.*, **48**, 1918.

Grouping the 30 radial velocities available for Cepheids according to distance, we get:

$\bar{r}$. parsecs.	$\bar{V}$. km./sec.	Number of Objects.	R. km.
196	14·8	5	$0{\cdot}9 . 10^{12}$
333	10·9	5	1·9
792	15·9	5	3·1
1360	14·8	6	5·7
2586	18·8	5	8·5
6030	16·6	4	22·5
1700	14·0	30	7·5

Even if the first three values be excluded, the progression in R is not favourable to Dr. Silberstein's views, and the conclusion seems to be that the spectral displacements of the Cepheids scarcely show any effect of the curvature of space-time.

(*b*) *The Novæ.*—The distances of the *Novæ* have been determined by the writer in several papers * on the reasonable assumption that the mean absolute maximum magnitude has a fairly constant value. The mean absolute magnitude at maximum, which evidently is more constant than the original or the final absolute magnitude of a Nova, has been derived by several different methods according to the following table:—

$\bar{M}_{max.}$	Weight.	
−6·2	2	Direct or indirect determinations of parallaxes for individual *Novæ*. Altogether 39 values for 11 objects have been used.
−6·5	1	From the "dip" of *Novæ*, caused by the location of our Sun being north of the galactic plane.
−8·2	1	From the concentration of *Novæ* in the central regions of our stellar system (Sagittarius-Scorpius), the distance of these being known from Shapley's work on globular clusters.
−4·2	½	From comparisons with the galactic distribution of other classes of stars.
−7·1	1	From proper motions and radial velocities (τ components)
−6·9	1	,, ,, ,, (v components).
−9·1	1	,, ,, ,, (parallactic motion).
−7·0±0·56		

The parallaxes of the 50 *galactic Novæ* known at present can be computed from the formula

$$\pi = 10^{-(2{\cdot}4+0{\cdot}2m_{max})},$$

where m_{max} is the apparent magnitude at maximum brightness.

* *P.A.S.P.*, 35, 95, 1923; 34, 207, 225, 1922: *Pop. Ast.*, 30, 621, 1922. The most extensive account of the writer's results concerning the determination of the distribution of Novæ in space is given in a paper in Swedish, published in the Journal of the Swedish Astronomical Society (*Popular Astronomisk Tidskrift*, H. 3–4, 1922).

The three last methods in the above-given table are liable to be in error, as no correction for a possible influence of the curvature of space-time has been applied to the radial velocities. But if we take the mean from the first four methods we find $\bar{M}_{max} = -6{\cdot}8$. As these methods do not imply any use of radial velocities for distant objects, we think the parallaxes of *Novæ* are free from any systematic error arising from the effect of the curvature of space-time.

As to the radial velocities for *Novæ*, reliable determinations exist for only 8 objects. The values given in the following table have as a rule been obtained from the five components of the H and K lines, but in three cases (*η Argus*, *T Coronæ*, and *P Cygni*) from several lines in the absorption spectrum. The very large displacements of the bright emission bands have not been considered to represent the motion of the star itself, and we think most astronomers agree with us in this view.

The reliability of the radial velocities thus used is illustrated by the fact that a solution for the apex gives $A = 17^h{\cdot}0$, $D = +12°$, $V_0 = -11{\cdot}72$ km./sec., or very close to the direction found for most classes of stellar objects. If we use the displacements of the emission bands the mean distance of *Novæ* would be 1,000,000 parsecs, a result which in itself seems incredible and which is not in harmony with other evidences mentioned above concerning the space-distribution of novæ.

There does not seem to be any progression in the radial velocities with the distance, but of course the material is meagre. The mean distance for 50 novæ is found to be 9600 parsecs, and, excluding two uncertain cases, 8600 parsecs. As the mean radial velocity is 13 km./sec., we obtain

$$R = 41 \cdot 10^{12} \text{ km.}$$

If we compute R from the individual values of V and r we obtain

	r. parsecs.	V. km./sec.	R. km.
Nova Persei 2	110	+ 6	$1{\cdot}1 \cdot 10^{12}$
Nova Aquilæ 3	320	−13	1·6
Nova T Coronæ	670	−10	4·1
Nova Geminorum 2	170	+10	1·1
Nova η Argus	625	−24	1·6
Nova Ophiuchi 4	7100	+ 6	73·0
Nova Cygni 1920	120	+17	0·4
Nova P Cygni	470	−17	1·7

The parallaxes of the objects *Nova Persei* 2, *Nova Aquilæ* 3, and *Nova T Coronæ* ought to be very accurate, as they have been determined by several independent methods; on the other hand, their distances are rather small for a determination of R, and therefore the value derived from the mean distance and the mean radial velocity is to be preferred.

(*c*) *The O Stars.*—These stars being giants or rather supergiants have been assumed to have a constant absolute magnitude. There are 50 stars for which the proper motion is known, and 27 for which the

radial velocity has been determined. Using these data for a determination of the mean parallax, we have found $\bar{\pi} = 0''{\cdot}00063$, corresponding to a value of $-4{\cdot}44$ for the absolute magnitude.

This value of the absolute magnitude may at first seem to be rather high, but it gets support from another method. In some cases there are involved in the galactic bright irregular nebulæ O stars as well as early B stars, and it seems to be a rule without exception that in such clusters of "nebulous" stars the O stars are considerably brighter than are the B stars. From eight such nebular clusters, where it can be

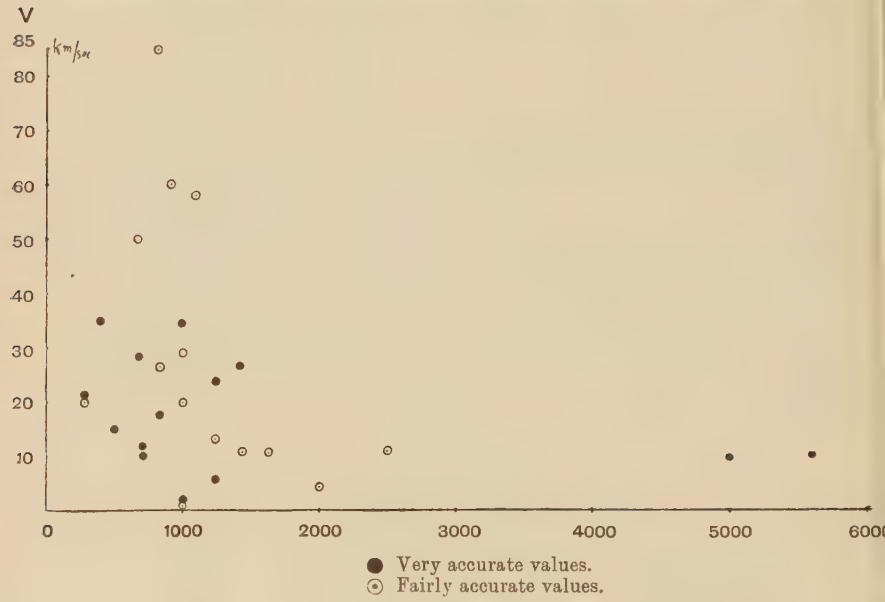


FIG. 3.—Radial velocities of O stars plotted against their distances.

tacitly assumed that the stars in question are practically at the same distance, it has been found:

$$\text{Abs. mag. for O stars} - \text{Abs. mag. for Bo stars} = -1{\cdot}74.$$

Adopting for the absolute magnitude of Bo stars $-2{\cdot}6$, in agreement with determinations of Charlier, Kapteyn, Adams, and others, we find the corresponding value for the O stars to be $-4{\cdot}34$. Combining the two values for the absolute magnitude, we see that the parallaxes of the O stars can be computed from the formula

$$\pi = 10^{-(1{\cdot}88+0{\cdot}2m)},$$

where m is the apparent visual magnitude.

No clear progression is indicated when grouping the radial velocities according to distance (fig. 3). As the individual values for the parallaxes

may be uncertain on account of the assumption of a small dispersion in the absolute magnitude,* we use, in order to determine R, the mean of the distances, or 1560 parsecs, compared with the mean radial velocity 24·2 km./sec. We find

$$R = 4{\cdot}0 \,.\, 10^{12} \text{ km.}$$

This is fairly close to the value computed by Silberstein, but it cannot have much significance, as the values of R computed from individual velocities will show a very large dispersion, and, besides, the most distant O stars will give a value for R around $200 \,.\, 10^{12}$ km.

It will be pointed out that the radial velocities are in many cases very accurate, as being determined from the orbits of spectroscopic binaries. The scattering in the R values mentioned above can by no means be explained either by the uncertainty in the measured Doppler

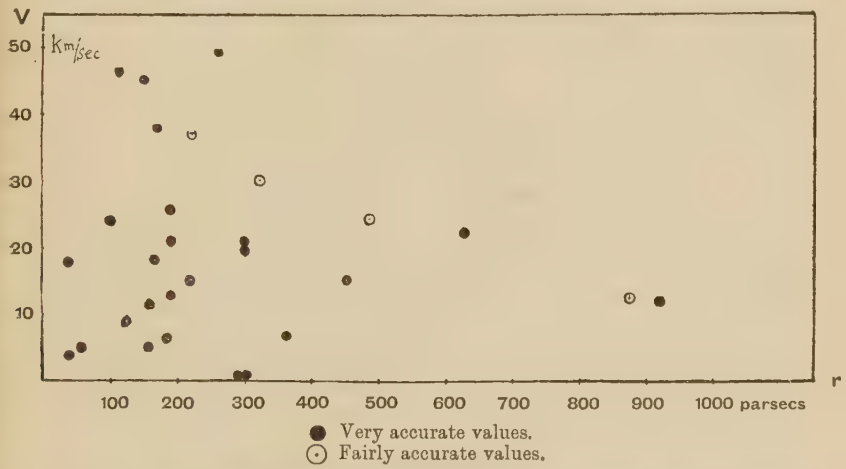


FIG. 4.—Radial velocities of eclipsing variables plotted against their distances.

shifts or by the uncertainty in the parallaxes derived from our assumption of negligible dispersion in the absolute magnitudes.

(*d*) *The Eclipsing Variables.*—The distribution in space of these stars has been obtained by Russell and Shapley † by using the surface intensity of the different spectral classes and certain elements from orbital determinations. The radial velocities are nearly all obtained from spectroscopic investigations of the orbit, and thus are very accurate.

There does not seem for these stars to be any progression in the radial velocities when plotted according to the corresponding distances (fig. 4). The mean value of 31 radial velocities is 18·2 km./sec., and the mean distance of the known variables is 800 parsecs.

Thus $R = 2{\cdot}7 \,.\, 10^{12}$ km.

* The dispersion in the absolute magnitudes seems to be fairly close to $\pm 1{\cdot}0$ magnitude, corresponding to a dispersion of about 45 per cent. in the distances.

† *Astrophys. Journ.*, **40**, 422, 1914.

As the distances of the eclipsing variables are comparatively small (all smaller than 1000 parsecs), it is probable that R cannot very well be determined from these stars, where the whole displacement may represent real motions. The fact that no progression is indicated in the radial velocities suggests that there is no effect of the curvature of space-time even in the distant stars of this group.

(*e*) *The R Stars.*—Accurate radial velocities for this interesting group of stars are due to the spectrographic work of Rufus* and of Sanford.† Using the known proper motions and radial velocities, Sanford found that the R stars are giants, as their mean absolute magnitude is $-1{\cdot}5$.

Using this value, distances for 29 stars have been correlated with Sanford's radial velocities according to the following condensed table:—

$\bar{r}$.	$\bar{V}$.	Number of Objects.	R.
parsecs.	km./sec.		km.
600	35·5	4	$1{\cdot}0 \cdot 10^{12}$
810	16·2	5	3·2
1130	114·5	6	0·6
1360	62·5	5	1·3
1720	34·1	5	3·1
2580	34·2	4	6·7

This case is really rather favourable for Dr. Silberstein's theory, but against the result stands the testimony from several other cases. Besides, the radial velocity for the R stars is seen to decrease with increasing distance, which suggests that if still fainter (or more remote) stars could have been investigated the R value would have been larger than $6{\cdot}7 \cdot 10^{12}$. Of course much weight cannot be attached to the first four values in the table, as the effect of the curvature may not show up for distances smaller than, say, 1000 parsecs.

(*f*) *The N Stars.*—There exist accurate radial velocities for 25 bright N stars, which have been determined by Moore,‡ who found that the absolute magnitudes for these stars must be high. The mean value was determined by him to be $-1{\cdot}1$ for the Na and $-2{\cdot}4$ for the Nb stars. The mean radial velocity is 18·8 km./sec., and as there does not seem to be any progression with distance, we adopt this value for the whole group. From Wilson's revision of the mean parallax it follows that the mean distance for the 92 N stars will be 710 parsecs, which gives the R value

$$2{\cdot}3 \cdot 10^{12} \text{ km.}$$

The dispersion in the radial velocities is rather large. Besides, Wilson§ is of opinion that the range in mean luminosity both in classes R and N makes estimations of distances of individual stars upon the assumption of equal luminosity within the class "decidedly uncertain."

* *Publications of the Observatory, University of Michigan*, **2**, 140, 1916.
† *Astrophys. Journ.*, **59**, 339, 1924; *Mount Wilson Contr.*, No. 276, 1924.
‡ *Lick Observatory Bulletin*, **10**, 160, 1922; *Publ. A.S.P.*, **35**, 124, 1922.
§ *Astr. Journ.*, **35**, 125, 1923.

As to other classes of red stars, extensive measures of the radial velocities for the M and S stars have been made by Merrill.* The mean parallaxes for the various subdivisions have been determined by Wilson,† Doig,‡ Strömberg and Merrill,§ and others. As the stars in question are comparatively near, the material has not been used for a determination of R. Furthermore, an uncertainty is involved by deriving parallaxes from the mean magnitudes, as the maximum brightness for the same long-period variable star does not seem to be a constant quantity.

Another group of stars that may be useful for testing Silberstein's theory are the so-called *c*-stars. These have not been used, as it is uncertain if all stars in the New Draper Catalogue said to have narrow sharp lines really can be included among the *bona-fide* *c*-stars.

(*g*) *The Spiral Nebulæ.*—The distances of these very interesting objects have been discussed during recent years from several points of view. Dr. Silberstein's statement that distances of 200,000 and 2800 parsecs have been derived by us for the Andromeda Nebula does not perhaps give an adequate expression to the views held in our papers of 1919 and 1920.‖ The different values given there for the Andromeda Nebula were deduced in order to show the state of the subject when approached in different ways. Besides, it was stated that the smaller distances were thought to be either limiting values or very uncertain, and that the largest one of 200,000 parsecs is probably the best. This value for the distance of the Andromeda Nebula was derived from the assumption that the absolute magnitude of *Novæ* at maximum brightness *in the mean* is the same for *Novæ* in the Galaxy and in the Andromeda Nebula. Later on it was adopted or confirmed by Curtis,¶ Luplau-Janssen and Haarch,** Doig,†† and others. From another method Öpik ‡‡ has derived a distance of 500,000 parsecs for the nebula. Later investigations by the writer suggest, on the basis of improved values for the absolute maximum magnitude of *Novæ*, a distance of the order of magnitude of half a million parsecs.§§

Since our first papers on the subject appeared it has been pointed out on several occasions |||| that the advances during later years in this branch have practically excluded the possibility of spiral nebulæ being as close to us as suggested by the parallax work of Bohlin ¶¶ or the theory of Jeans.*** The mean distance of spirals has at least to be

* *Mount Wilson Contr.*, Nos. 264, 265, 1923; *Astrophys. Journ.*, 58, 215, 1923.
† *Astr. Journ.*, 34, 183, 1923.
‡ *Journ. B.A.A.*, 33, 326, 192, 1923.
§ *Mount Wilson Contr.*, No. 267; *Astrophys. Journ.*, 59, 97, 1924.
‖ *Astr. Nachr.*, 209, 369, 1919; *Kungl. Sv. Vet. Akad. Handl.*, Bd. 60.
¶ Discussion with Shapley published as the pamphlet, *The Scale of the Universe*, also *Rivista di Scienza*, 35, No. 1, 1924.
** *Astr. Nachr.*, 215, 285, 1922.
†† *Journ. of the B.A.A.*, 32, 138, 1922.
‡‡ *Astroph. Journ.*, 56, 406, 1922.
§§ *Publ. A.S.P.*, 35, 95, 1923.
|||| See, for instance, *Publ. A.S.P.*, 34, 108, 1922.
¶¶ Bohlin measured a value for the parallax of the Andromeda Nebula of $0''\cdot171 \pm 0''\cdot051$, which later has been reduced to $0''\cdot051 \pm 0''\cdot079$. Other observers have got much smaller values (for instance, Barnard and van Maanen).
*** *Problems of Cosmogony and Stellar Dynamics*, Cambridge, 1919.

counted in several ten-thousands of light-years, and thus it is of quite another order of magnitude than the mean distance of the stars in the regions where most of the spiral nebulæ have been observed. On the other hand, the closest spirals cannot very well be situated at distances of many millions of light-years as originally suggested by Curtis,* inasmuch as several of the largest spirals are in their outer regions resolved into stars (*e.g.* Messier 33), and in accordance with our present knowledge ordinary stars as a rule very seldom have an absolute magnitude higher than -7 or -8. Besides, *Novæ* have been observed in seven spiral nebulæ, variable stars in one, and bright diffuse nebulosity in a few, and our knowledge concerning the physical properties of these objects prevents us placing the ones observed in spirals as far away as millions of light-years. From the measures of the effective wavelength of some 40 spirals, Lundmark and Lindblad † concluded that there was no sensible selective absorption in space, and at the same time Shapley ‡ arrived at the same conclusion from his cluster work.

8. Jeans § has recently computed distances corresponding to parallaxes of 0″·0007, 0″·0011, and 0″·0066 for the spiral nebulæ M32, M101, and M51 respectively. In the last case the parallax is really so large that it ought to be possible to measure it by trigonometric methods. Although Dr. van Maanen ‖ has by direct measurement found a parallax of 0″·005 for this object, I do not think he considers this circumstance as actually proving the nebula to be so close.¶ Besides, the proper motion of the object is small, and the internal motions suggest a larger distance than this. The method of Jeans for finding the parallaxes of nebulæ is very ingenious, but it does not give very reliable information about the real distances. The method is based upon a calculation of the mean distance between adjacent nuclei in the spiral arms. The comparison of these calculated distances with the mean apparent distances as estimated on photographs then gives the parallax. Taking the theory for granted, one can still question if it is possible to get a fair estimate of the apparent distance of nuclei. For instance, Jeans finds the nuclei in M101 to be 5″ apart. But a much larger instrument than the 100-inch at Mount Wilson would undoubtedly reveal many more nuclei and show them to be apparently much closer than is the case on the photographs of to-day.** The so-called nebulous background in the spirals is as a rule certainly not nebulosity, but the light from numerous minute nuclei apparently gathered together on account of the vast distance from which they are seen. Jeans' estimates show us that the objects probably *are not closer* than estimated by him, but that they might very well be farther away. His method will be of value in the future, when

* *Modern Theories of the Spiral Nebulæ*, Washington, 1919; *Publ. A.S.P.*, 29, 206, 1917; *Lick Obs. Bull.*, No. 300, 1917.

† *Astroph. Journ.*, 46, 206, 1917, and also 50, 376, 1919.

‡ *Mount Wilson Contr.*, 156, 1918.

§ *The Nebular Hypothesis and Modern Cosmogony*, Halley Lecture, 1922, Oxford, 1923.

‖ *Mount Wilson Contr.*, 158, 1918. ¶ van Maanen, *op. cit.*, p. 7.

** We may compare the photographs of Isaac Roberts and Ritchey of the same spirals. Where the former show a nebulous appearance the latter in many cases show numerous nuclei.

we know for certain that we have succeeded in completely resolving the spirals into stars.

One of the methods most free from objection is to derive the distances from comparisons between the proper motions and radial velocities. From available measures of the position of the central nuclei proper motions have been derived in Upsala for 82 spiral nebulæ.* The mean annual proper motion is found to be 0″·042. The motions thus derived are certainly not real. Computing the apex by using Airy's method, we find:

$$A = 19^{h}\cdot 2$$
$$D = -69^{\circ}$$
$$q = 0''\cdot 0135.$$

The deviation of this apex-value from the one found from radial velocities, and earlier referred to in this paper, suggests that our proper motions are strongly influenced by systematic errors. A detailed comparison with the proper motions derived by Curtis for 66 spirals from measures of photographs has made us believe that our motions are illusory in consequence of the influence of accidental errors in the measures, and only of use for giving an upper limit to the motion, as the real mean motion will be equal to or smaller than the mean error of the mean motion.

Using the value of 793 ± 88 km./sec. for Campbell's K term, we find the following values for the mean parallax of spiral nebulæ:—

π_m.	Material.	Number of Objects.
0·00010	Curtis	66
0·00011	Lundmark	82
0·000043	van Maanen	7
0·000054	,, (Smart's values)	7

If we had assumed $K = 0$ the mean parallaxes would not have been materially changed.

As stated before, the values of van Maanen are certainly the most accurate of the existing proper motions. Still it is quite possible that the actual motions are smaller, and that even the last value in the table represents an upper limit for the mean parallax of the spiral nebulæ investigated.

The mean radial velocity of the five of van Maanen's nebulæ for which this quantity has been measured is 220 km./sec., and the mean radial velocity for the nebulæ of Curtis and Lundmark will be around 600 km./sec. Computing R according to Silberstein's formula, we have:

R. km.	Material.
$1\cdot 0 \cdot 10^{12}$	Curtis
$0\cdot 9 \cdot 10^{12}$	Lundmark
$6\cdot 6 \cdot 10^{12}$	van Maanen
$5\cdot 2 \cdot 10^{12}$	,, (Smart's values)

* *Pop. Astr.*, 30, 623, 1922; also read at Twenty-eighth meeting of the Amer. Astr. Soc., Williams Bay, 1922.

The very close agreement between the two last values and the one adopted by Silberstein is at first very striking, and it is tempting to take the result as confirming Silberstein's theory. But such a comparison would be premature, for the following two reasons. First, there are, as mentioned above, reasons to believe that the mean distance derived by this method is too small; and, second, in deriving the parallax we have used the Doppler shifts measured in spectrograms of spiral nebulæ as indicating true motions. But as the effect of the curvature of space-time will certainly be present in the spectra of spiral nebulæ in still higher degree than in the globular clusters, our procedure, which involves the supposition that the spectral shifts represent radial motions only, may well lead to wrong ideas about the distances.

If we want to investigate the relation between V and r for the spiral nebulæ measured for Doppler shift, we can make use of the hypothetical parallaxes given in Table III. The values have been founded

TABLE III.

Hypothetical Parallaxes, Relative Distances, Radial Velocities, and Relative R Values for Spiral Nebulæ. a is a Scale Factor.

N.G.C.	Parallax. From Total Magnitude.	Parallax. From Apparent Diameter.	Distance. The Unit is the Dist. of Andromeda Neb.	Radial Velocity.	Relative Values of R.
221 }	0″·00000510	0″·00000510	1·0	− 300	1·00a
224 }	510	510	1·0	− 316	0·95
278	3	6	130	+ 650	60·0
404	7	5	88	− 25	1056·0
584	8	7	68	+1800	11·3
598 }	225	230	2·2	− 70	9·4
604 }	225	230	..	− 270	..
936	6	13	62	+1300	14·3
1023	14	25	28	+ 200	42·0
1068	70	25	14	+1020	4·1
1700	2	3	212	+ 800	79·5
2681	6	4	106	+ 700	45·4
2683	16	29	25	+ 800	9·4
2841	35	29	16	+ 600	8·0
3031	105	75	5·8	− 30	58·0
3034	50	35	12	+ 290	12·4
3115	31	22	20	+ 600	10·0
3368	16	14	34	+ 940	10·8
3379	35	8	39	+ 825	14·2
3489	6	10	68	+ 600	34·0
3521	15	25	27	+ 730	11·1

TABLE III.—*continued.*

N.G.C.	Parallax. From Total Magnitude.	Parallax. From Apparent Diameter.	Distance. The Unit is the Dist. of Andromeda Neb.	Radial Velocity.	Relative Values of R.
3623	0″·00000017	0″·00000033	23	+ 800	8·6
3627	55	33	12	+ 650	5·6
4111	11	10	49	+ 800	18·3
4151	4	13	83	+ 940	26·5
4214	4	29	72	+ 300	72·0
4258	90	75	6·2	+ 500	3·7
4382	16	16	32	+ 500	19·2
4449	31	19	22	+ 200	33·0
4472	16	12	37	+ 850	13·1
4486	24	12	32	+ 800	12·0
4526	6	19	56	+ 580	29·0
4565	7	62	40	+1100	10·9
4594	50	29	14	+1140	3·7
4649	29	8	40	+1090	11·0
4736	95	23	14	+ 290	14·5
4826	45	33	13	+ 150	26·0
5055	29	34	16	+ 450	10·8
5005	4	21	76	+ 900	25·3
5194	106	50	7·5	+ 270	8·3
5195	106	50	7·5	+ 230	(9·8)
5236	31	42	14	+ 500	8·4
5866	4	11	87	+ 650	40·2
7331	0″·00000011	0″·00000038	30	+ 500	18·0

on the above-mentioned value of 200,000 parsecs for the Andromeda Nebula and derived from the supposition that the apparent angular dimensions and the total magnitudes of the spiral nebulæ are only dependent on the distance. Of course this is a rather rough hypothesis, but probably the best we can use for the present. The parallaxes derived from the total magnitudes have been reduced to the scale of those derived from the apparent diameters. The total magnitudes used have generally been estimated by Holetschek and revised by Hopman. The dimensions of the objects have been measured on plates from the Crossley collection of the Lick Observatory.

An inspection of the Table III. will show that a computation of R from the individual values of V will give inconsistent values for the radius of curvature.

From the parallaxes the mean distances have been calculated, the distance of the Andromeda Nebula being taken as the unit. Plotting the radial velocities against these relative distances (fig. 5), we find that there

may be a relation between the two quantities, although not a very definite one. If this phenomenon were due to the curvature of space-time, we could derive the mean linear distance or determine the scale of our relative distances in the following way:—

Assuming that the average $\overline{v_0}^2$ for a group of objects is the same as

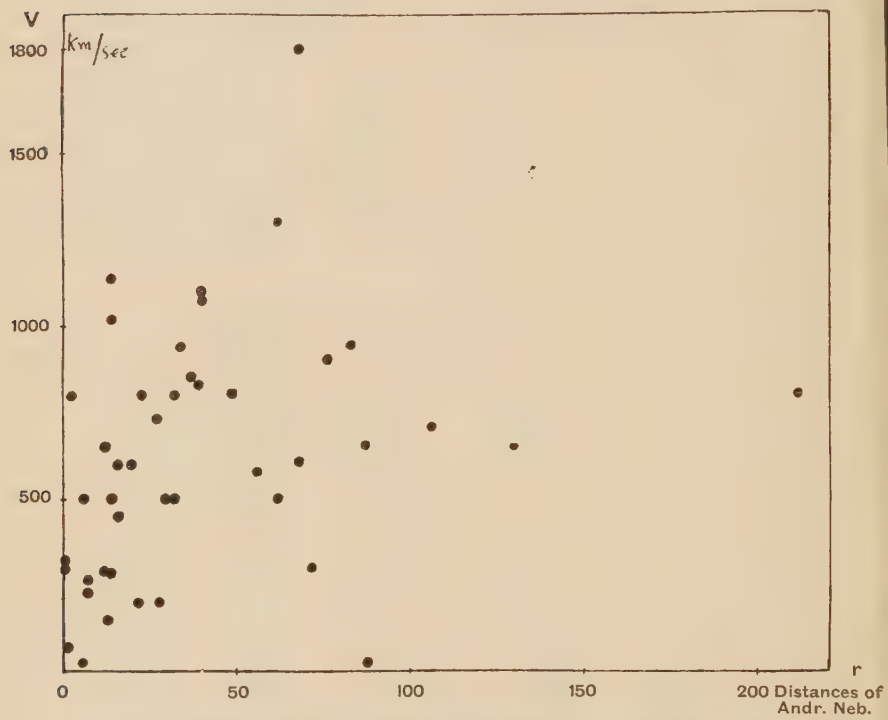


FIG. 5.—Relation between the relative distances (the unit is the distance of the Andromeda nebula) and the measured radial velocities of spiral nebulæ.

for another group, placed at another distance, Silberstein has from the reduced Doppler effect deduced the formula

$$\overline{D_1}^2 - \overline{D_2}^2 = \frac{1}{R^2}(\overline{r_1}^2 - \overline{r_2}^2),$$

where D are the Doppler shifts and r the distances. [The bars denote the average values and the suffixes two different groups of objects.]

Taking two groups of spiral nebulæ we could compare them, for instance, with a group of globular clusters. Denoting the quantities of spirals and globulars with suffixes s and g, and eliminating R, we obtain

$$\frac{\overline{r_s}^2 - \overline{r_g}^2}{\overline{D_s}^2 - \overline{D_g}^2} = \frac{\overline{r_{s_1}}^2 - \overline{r_{s_2}}^2}{\overline{D_{s_1}}^2 - \overline{D_{s_2}}^2}.$$

In our table we have not r_s but the relative values ρ_s of the spiral distances. Putting

$$r_s = \rho_s \epsilon,$$

where ϵ is the scale-factor to be determined, we find the following equation for ϵ:

$$\epsilon^2 = \frac{\overline{r_g^2}(\overline{D_{s1}^2} - \overline{D_{s2}^2})}{\overline{\rho_s^2}(\overline{D_{s1}^2} - \overline{D_{s2}^2}) - (\overline{\rho_{s1}^2} - \overline{\rho_{s2}^2})(\overline{D_s^2} - \overline{D_g^2})}.$$

If the dispersion of the individual values of r or ρ within a group is small compared with the mean distance itself, then the formula above can be written

$$(\overline{D_1})^2 - (\overline{D_2})^2 = \frac{1}{R^2}\left(\bar{r}_1^2 - \bar{r}_2^2\right).$$

Grouping our material in the table according to the size of radial velocity, we find

$\bar{V}$. km./sec.	$\bar{r}$.	No. of Objects.
386	$30{\cdot}8 \pm 32{\cdot}0$	23
963	$54{\cdot}5 \pm 47{\cdot}0$	18

As the dispersion in r is of the same order of magnitude as $\bar{r}$ itself, the application of the simplified formula is not justified in this case.

Computing ϵ according to the rigorous formula, we find that its value or the linear distance of the Andromeda Nebula is

36,000 parsecs.

The value of R is $2{\cdot}4 \cdot 10^{12}$ km., and if we use the Andromeda Nebula alone we find it to be $6{\cdot}6 \cdot 10^{12}$ km.

If this distance of the nebula is correct, the parallaxes in Table III. should be multiplied by a factor close to 6. On account of the rather obscure correlation between V and r, not much weight can be attached to this determination of the distance of the Andromeda Nebula. Besides, the presumption that $\overline{v_0^2}$ has the same value for the globular clusters and the spiral nebulæ certainly is open to objection.

Dr. Silberstein's above-quoted formula, in which $\overline{v_0^2}$ is eliminated, will certainly be practicable for determining R if rather near objects are used. The best objects for such an application ought to be the moving clusters and some of the open clusters, because in several cases their parallaxes and radial velocities are known with very high accuracy. Besides, such groups are in much the same physical state, and the assumption of $\overline{v_0^2}$ being equal is probably fairly justified.

As the dispersion in the distances and in the Doppler shifts of the individual stars in a cluster is small, we can use the simplified formula for the computation.

The following list of objects has been selected :—

Object.	$\bar{r}$.	$\bar{V}$.
	"	km./sec.
Taurus group	0·0275	+36·4
Coma Berenices	0·012	−. 2·0
Pleiades	0·0080	+10·0
Præsepe	0·0068	+33
Orion cluster	0·0036	+21·4
Messier 67	0·0013	+13
h and χ Persei	0·0007	−43·2
N.G.C. 1647	0·0022	+66
N.G.C. 6709	0·0012	+15

If we first compare the *Taurus* group with the more distant clusters, we see that R in several cases turns out *imaginary*, since the formula demands if $r_1 > r_2$ that $D_1 > D_2$. The same is the case, with two exceptions, if we compare *Præsepe* with more distant clusters, and, taking together the combinations where we get real values for R, we see that R cannot be determined at least from our present material with any accuracy.

Greenwich :
1924 *August.*

Naked-Eye Observations of Venus.

By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

Recent work has assigned a new importance to the ancient observations of the planet Venus, and modern observations of the times when Venus is first seen and last seen near the Sun may be of value in interpreting the ancient ones, even though the climatic conditions may be seriously different. The following observations have been put into my hands, and I take full responsibility for publishing them, to which the observers are somewhat reluctant, as perhaps implying too great an importance. But it seems to me that they may be useful to others as a guide at any rate, and perhaps a stimulus : and with the above disavowal plainly stated, publication cannot do much harm.

All observations were made with the naked eye, except that J. K. F. wore ordinary eye-glasses. They were all made at Oxford except the last. No places of Venus were computed in advance. All altitudes are true, refraction being disregarded.

First Appearance in Evening.

1923 Nov. 4. J. H. J. watched from $4^h\ 35^m$ to $4^h\ 55^m$ G.M.T. Clear but slight mist. Venus not seen.

5 and 6. Cloudy.

7. Venus picked up at $4^h\ 33^m$; very difficult; became quite easy by $4^h\ 40^m$; disappeared in mist between $4^h\ 45^m$ and $4^h\ 50^m$, say $4^h\ 47^m{\cdot}5$.

Computed Altitudes.

	Time.	Alt. of Venus.	Alt. of Sun.
	h m	°	°
Venus seen	4 33	+3·5	−1·4
Venus quite easy	4 40	+2·4	−2·4
Disappeared	4 47·5	+1·5	−3·5
Venus set	4 59·2	...	−5·20

Stellar magnitude of Venus −3·4.

Last Appearance in Evening.

1924 June 16. Cloudy.

17. Rather cloudy, then lightning and heavy rain.

18. Rather cloudy. Venus not seen.

19. Venus seen by J. K. F. at $8^h\ 40^m$ or $8^h\ 41^m$ G.M.T., but lost at $9^h\ 8^m$. Quite conspicuous.

20. J. K. F. and Miss E. D. observed. Venus seen by E. D. at $8^h\ 40^m$, by J. K. F. at $8^h\ 47^m$. Lost by both at $9^h\ 7^m$. J. K. F. found Venus very faint, and believes she would not have been found by him had he not known from the previous night's observation her exact position.

21. J. K. F., S., and A. M. F. observed. The lowest part of the horizon was strongly illuminated by the Sun, and there was a cloud-band above it. Venus was not seen.

22. Cloudy generally, especially in low N.W. Venus not seen. J. K. F., S., and Miss P. observed.

The computed altitudes for June 20 are:

	Time.	Alt. of Venus.	Alt. of Sun.
	h m	°	°
Venus seen	8 40	+4·8	−2·3
Venus lost	9 7	+1·2	−5·2
Venus set	9 16·6	0·0	−6·11

Stellar magnitude of Venus −3·5.

First Appearance in Morning.

All observations made from Radcliffe Observatory. J. K. F. and S. observed daily: A. M. F. on July 11.

1924 July 10 (astron.). Clear. Venus not seen.

11 ,, Hazy. ,, ,,

12 ,, Cloudy. No observation attempted.

13 ,, Clear. J. K. F. saw Venus at $15^h\ 29^m{\cdot}5$ G.M.T., and pointed her out to S.: J. K. F. kept her in view till $15^h\ 53^m{\cdot}5$, S. till $16^h\ 4^m$. Venus clearly visible.

Computed Altitudes.

	Time.	Alt. of Venus.	Alt. of Sun.
	h m	°	°
Venus rose	15 11·9	...	−6·50
Venus seen	15 29·5	+2·4	−4·6
Venus lost	16 4	+7·3	−0·6

Stellar magnitude of Venus −3·7.

J. K. F. notes his surprise to find the planet visible so long at first and last appearance. He had expected it to be for a few minutes only, say seven, and rather high in the sky.

On the morning of July 13, H. G. T. made a slightly earlier observation at Dedham (1° 0′ E., 51° 57′ N.).

	Time.	Alt. of Venus.	Alt. of Sun.
	h m	°	°
Venus rose	15 2·2	...	−6·42
Venus seen	15 18	+2·1	−4·8

The most striking observation up to the present is thus that of J. H. J. on 1923 Nov. 7.

University Observatory, Oxford:
1924 *September* 6.

Occultation of B.D. −15°6169 *by Mars observed on* 1924 *June* 21 *with the* 243-*mm. Merz-Repsold Refractor at the Astronomical Observatory, Kiew.* By Professor S. D. Tscherny.

I. *Results of Micrometer Measures.*

Before the occultation of B.D. −15°6169 by Mars, I measured on June 21 with the filar micrometer (eyepiece with a magnifying power 145) the angle of position (p) and the distance (s) of this star from the centre of Mars. The results of my micrometrical measures are as follows:—

h m s			° ′
19 18 41·0	K.S.T.	p =	73 41·25
19 45 53·5	,,	p =	69 44·25
19 29 23·0	,,	s =	1″·478
19 39 6·75	,,	s =	1″·166,

or

19 32 17·25	,,	p =	71° 42′·75
19 34 14·87	,,	s =	1″·322.

Reducing the distance to the moment $19^h\ 32^m\ 17^s{\cdot}25$ K.S.T. = $11^h\ 30^m\ 46^s{\cdot}10$ G.M.T., we have

$$p = 71^\circ\ 42'{\cdot}75\,; \quad s = 1''{\cdot}385.$$

The value of one revolution of the micrometer screw at the temperature when the observations were made (+17°·5 R.) is 28″·2568. Therefore, for the moment 11ʰ 30ᵐ 46ˢ·10 G.M.T. we have

	p	s
Differential refraction	71° 42′·75	39″·136
	− 3′·17	+ 0″·012
	71° 39′·58	39″·148

Based upon these results of observations, we have

$$\alpha_* - \alpha_\delta = +38''\cdot56 = +2^s\cdot57\,; \quad \delta_* - \delta_\delta = +12''\cdot3,$$

where *

$$\alpha_* = 22^h\ 10^m\ 32^s\cdot57\,; \quad \delta_* = -15^\circ\ 28'\ 26''\cdot5.$$

Parallax of Mars:

$$\alpha_{app.} - \alpha_{geoc.} = +0^s\cdot42\,; \quad \delta_{app.} - \delta_{geoc.} = -12''\cdot9.$$

Therefore, for 11ʰ 30ᵐ 46ˢ·10 G.M.T., the geocentric co-ordinates of Mars were

$$\alpha = 22^h\ 10^m\ 29^s\cdot58\,; \quad \delta = -15^\circ\ 28'\ 25''\cdot9.$$

For the same time we have (*a*) from *C. d. T.*:

$$\alpha = 22^h\ 10^m\ 29^s\cdot68\,; \quad \delta = -15^\circ\ 28'\ 23''\cdot3.$$

The correction to Le Verrier's Tables of Mars is

$$\Delta\alpha = -0^s\cdot10\,; \quad \Delta\delta = -2''\cdot6.$$

(*b*) From *N.A.* or *B.A.J.* the position of Mars is

$$\alpha = 22^h\ 10^m\ 29^s\cdot46\,; \quad \delta = -15^\circ\ 28'\ 25''\cdot0.$$

The correction of the Newcomb's Tables, with Ross's corrections, is

$$\Delta\alpha = +0^s\cdot12\,; \quad \Delta\delta = -0''\cdot9.$$

II. *Occultation of B.D.* − 15°6169 *by Mars.*

During the occultation the position of Mars was near the horizon ($h = 20^\circ$), and therefore an eyepiece of magnifying power only 145 was used. Near Mars the sky was lighted by the moon in the last quarter. As the star approached the bright disk of Mars its brightness decreased. The observation of disappearance was very difficult, because the star appeared attached to the bright limb of Mars. This phenomenon probably depended on the circumstances of the atmospheres of Mars and of the earth.

The results of my observations are as follows:—

h	m	s		
12	6	7·04	G.M.T.	The star appeared attached to the bright limb of Mars.
12	6	31·98	„	Doubtful if the star had disappeared.
12	6	42·95	„	*Disappearance.*
12	24	59·94	„	The star reappeared instantaneously, though faint, at the dark limb of Mars.

* L. J. Comrie, "Planetary Occultations in 1924," *Journal. of B.A.A.*, **34**, No. 6, p. 239.

After the reappearance of the star I measured with the filar micrometer the position-angle of the star from the centre of Mars and obtained for $12^h\ 30^m\ 35^s{\cdot}03$ G.M.T.

$$p = 280^\circ\ 22'{\cdot}0.$$

From the times of disappearance and reappearance, the co-ordinates of Mars (*N.A.*), semidiameter of Mars $= 7''{\cdot}95$ and hor. par. $= 14''{\cdot}85$, I deduced the following equations of condition:—

$$\begin{array}{ll} \text{D.} & [1{\cdot}89821]\ \Delta\alpha + [1{\cdot}70771]\Delta\delta = [2{\cdot}30875]'' \\ \text{R.} & [1{\cdot}78430]_n\Delta\alpha + [1{\cdot}34328]\Delta\delta = [1{\cdot}75218]_n''. \end{array}$$

The solution of these equations gives the following values:—

$\Delta\alpha = +1''{\cdot}520 = +0^s{\cdot}101$; $\Delta\delta = +1''{\cdot}633$ for Newcomb's Tables, with Ross's corrections, and

$\Delta\alpha = -0^s{\cdot}12$; $\Delta\delta = -0''{\cdot}1$ for the Tables of Le Verrier (*C. d. T.*).

Astronomical Observatory, Kiew:
1924 *June* 26.

Observations of the Transit of Mercury on 1924 *May* 7.*

(*Communicated by the Secretaries.*)

The following observations were made at the Sydney Observatory:—
The sky was overcast at time of ingress, but good observations at meridian transit and egress were obtained.

Transit Circle Observations.

Mercury was close to the Sun's centre at noon, and the observations comprised:—

Time of transit of Sun's preceding limb.
Circle reading of Sun's north limb.
Transit and circle reading of Mercury.
Circle reading of Sun's south limb.
Time of transit of Sun's following limb.

The series happens to be so symmetrical that it was not considered necessary to apply any differential corrections.

A handle had previously been fitted to the slow motion screw in order to bring the telescope rapidly into the successive positions required, and all circle settings were made by means of the tangent screw, keeping the declination micrometer screw fixed throughout. The observers were the regular transit observers, Messrs. Raymond and Cranney, who rehearsed the series for several days preceding the phenomenon with imaginary transits and settings for the planet. The following results were obtained:—

* For observations previously published see p. 543 *et seq.*

Clock times of transit:—

	h	m	s
Sun's preceding limb	2	57	00·19
Sun's following limb	2	59	13·41
Sun's centre	2	58	06·80
Mercury	2	58	01·47
Mercury precedes Sun			5·33

Circle readings:—

	°	′	″
Sun's north limb	100	22	29·2
Sun's south limb	100	54	13·4
Sun's centre	100	38	21·3
Mercury	100	37	21·6
Mercury north of Sun			59·7

	h	m	s
G.M.T. of Sun's transit May 7	13	51	36·48

Observations with the Equatorial.

Mr. Rayner used the 11¼-inch, with a polarising solar eyepiece, and a sidereal chronograph. Messrs. Raymond and Cranney used a projection of the Sun, with an M.T. chronometer. The two observations were quite independent. The Sun's limb was boiling, but the times agree well with one another. External contact was considered worthless.

G.M.T. of internal contact:—

	h	m	s
Mr. Rayner . . . May 7	17	34	19·2
Messrs. Raymond and Cranney	17	34	19·4

Mr. Rayner's observations would be the more reliable of the two, and so his result is adopted. The time is $39^{s}\cdot8$ sooner than that predicted according to *Nautical Almanac* figures.

Taking the relative rate of movement of sun and planet, as given in the *Nautical Almanac*, viz. $14^{s}\cdot98$, this would indicate an error of $0^{s}\cdot16$ in their relative R.A., thus agreeing very closely with transit observations, which show an apparent error of $0^{s}\cdot15$.

The following observations were made at the Hector Observatory, Wellington, New Zealand:—

Owing to clouds the ingress at $9^{h}\ 45^{m}\ 31^{s}\cdot5$ was not observed. The Sun was not visible until $11^{h}\ 5^{m}$. From then until sunset at $16^{h}\ 49^{m}\ 33^{s}$ the Sun was visible through passing clouds. At meridian transit the sky was clear and observations were made by C. E. Adams and R. C. Hayes.

Both observations were made by projecting the Sun's image on screens attached to the transit instruments.

Transit Instrument No. 1 is 3 inches aperture and 36 inches focal length, and an eyepiece giving 110 magnification was used. Seven measures of the distance between the preceding limb of the Sun and the centre of Mercury were made. The telescope was then reversed, and eight measures of the distance between Mercury and the Sun's following limb were made. The observer was Mr. C. E. Adams. The results were:—

	m	s	
Sun's *p* limb to Mercury . .	= 1	24·83 ± ·09	S.T.
Mercury to Sun's *f* limb . .	=	48·25 ± ·06	S.T.

Hence S.T. of Sun's semi-diameter passing the meridian = $1^m\ 6^s{\cdot}54$, and Sun to Mercury = $18^s{\cdot}29$.

	h	m	s	
G.M.T. of conjunction . . .	= 13	30	47	*N.A.*, p. 469.
Sun on meridian at Hector Observatory =	12	17	22·06	
	1	13	24·94	

Using the hourly motions given in the *Nautical Almanac*, the calculated difference between the Sun and Mercury is $18^s{\cdot}33$. This differs $0^s{\cdot}04$ from the observed.

Transit Instrument No. 2 is $2\frac{3}{4}$ inches aperture and 33 inches focal length, and eyepiece giving 40 magnification was used. Five measures of the distance of Mercury from both limbs of the Sun were made. The instrument was not reversed during the observation. The observer was R. C. Hayes. The results were:—

	m	s	
Sun's *p* limb to Mercury . . .	= 1	24·99	S.T.
Mercury to Sun's *f* limb . . .	=	48·50	S.T.

Hence S.T. of Sun's semi-diameter passing meridian = $1^m\ 6^s{\cdot}74$, and Sun to Mercury = $18^s{\cdot}23$.

	h	m	s
G.M.T. of conjunction . . .	= 13	30	47
Sun on meridian at Hector Observatory =	12	17	22·06
	1	13	24·94

Using the hourly motions given in the *Nautical Almanac*, the calculated difference between the Sun and Mercury is $18^s{\cdot}33$, differing $0^s{\cdot}10$ from the observed.

Errata.

P. 99. *Micrometrical Measures of Double Stars: for*

	h	m	°	′	
Σ3056	0	0·8	+33	52	7·4, 7·4,
read Σ3058	0	1·5	+29	55	7·7, 9·2.

Plate 15, *for* 2-in. Pris. Cam. *read* 12-in Pris. Cam.

P. 603, last line, *for* $\frac{31}{60480} f_4(n)$ *read* $\frac{31}{60480} f_6(n)$.

LISTS OF ADDITIONS

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JUNE 1912 *TO JUNE* 1924

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1924

Printed in Great Britain by
Neill & Co., Ltd., Edinburgh.

LIST OF ADDITIONS

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1923 JUNE 30 TO 1924 JUNE 30.

Serial Publications received in the regular course, and Reprints of Papers published in some of these Serials, are not included in this Annual List.

An asterisk (*) indicates that the work is an excerpt.

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Geschichte des Fixsternhimmels, enthaltend die Sternörter der Kataloge des 18 u. 19 Jahrhunderts. Abteilung 1. Der nördliche Sternhimmel. Bänder 1, 2.
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The Theory of Spectra and Atomic Constitution.
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Memoirs. Vol. 14, No. 4.
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*Über die Verteilung der Sterne verschiedener Massen in den kugelförmigen Sternhaufen.
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OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXXIV. NO. 1. NOVEMBER 1923.

PUBLISHED BY THE ROYAL ASTRONOMICAL SOCIETY,
BURLINGTON HOUSE, LONDON, W. 1.

AND SOLD BY

WHELDON & WESLEY, LTD.

2, 3, AND 4 ARTHUR STREET, NEW OXFORD STREET,
LONDON, W.C. 2.

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Printed in Great Britain by NEILL & Co., LTD., Edinburgh. December 1923.